Introducing Box Chains to simplify Reachability Analysis

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Summer Workshop on Interval Methods, 2024

Avec le soutien de

Introductive Problem

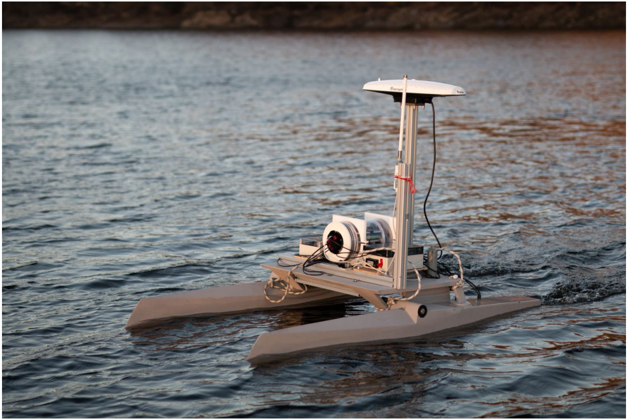
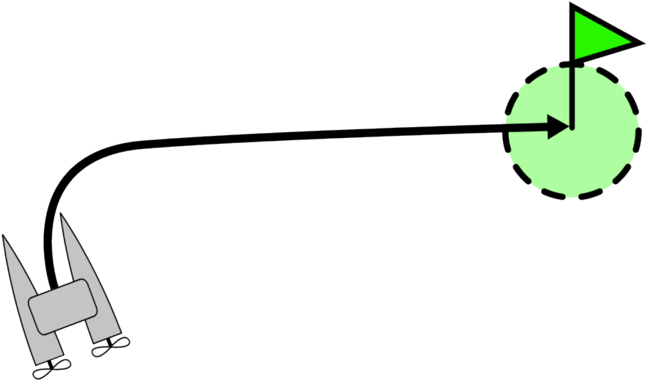


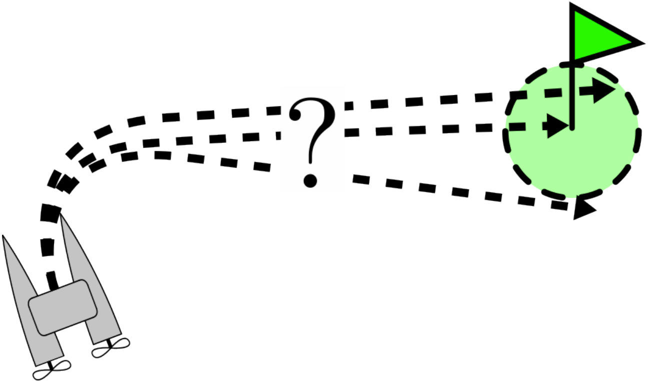
Figure: Helios

Introductive Problem

Perfect case



Perturbated case



Definition

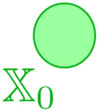
Definition (Reachable set at a point in time)

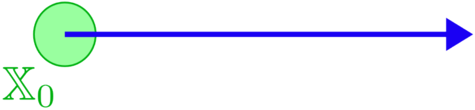
Consider a dynamical system following a state equation of the form $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$. The set of initial states and inputs are bounded i.e $\mathbf{x}(0) \in \mathbb{X}_0$ and $\mathbf{u} \in \mathbb{U}$. The reachable set at a certain point of time tr is defined as the union of the possible system states at $t = t_r$:

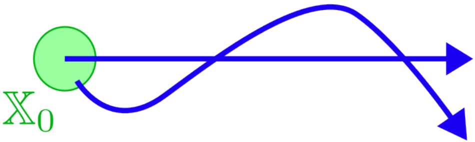
$$\mathcal{R}(t_r) = \left\{ \mathbf{z} \in \mathbb{R} | \int_0^{t_r} \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) dt, \mathbf{x}(0) \in \mathbb{X}_0, \mathbf{u}\left[0, t_r\right] \in \mathbb{U} \right\}$$

where $\mathbf{u}\left[0, t_r\right] = \bigcup_{t \in [0, t_r]} \mathbf{u}(t)$

Illustration




$\mathbb{X}_0$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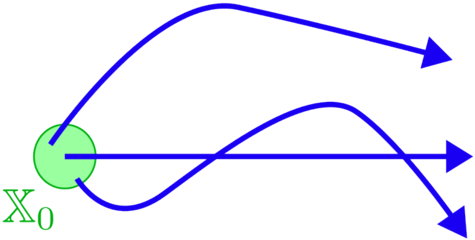
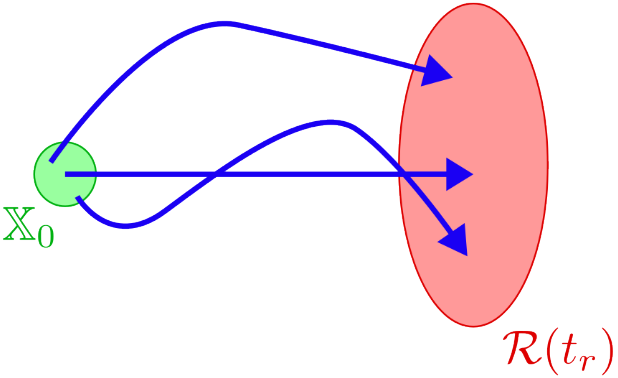

$\mathbb{X}_0$



$\mathbb{X}_0$
$\mathcal{R}(t_r)$

Starting result

From [1] (Thomas Lew 2023) applied with Interval Analysis tools

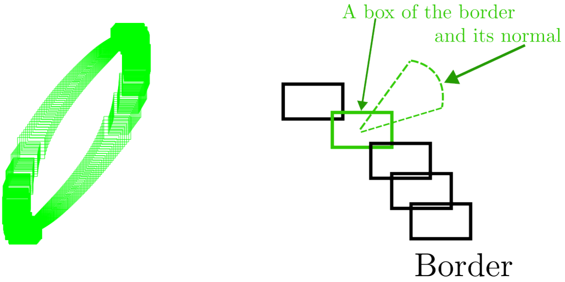


Figure: Frontier of the Reachable Set

Each **box** is the result of a **guaranteed integration**

Self-intersecting frontier

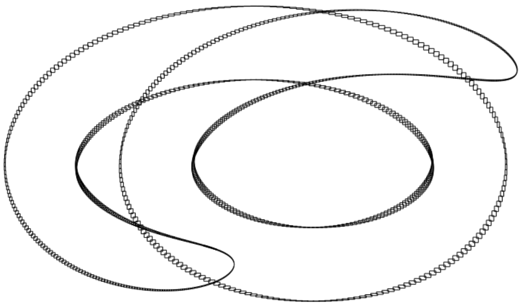


Figure: Self-intersecting frontier

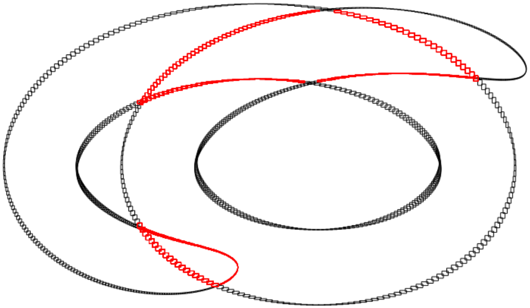


Figure: Fake boundaries

Context

We define two functions :

- $\mathbf{f} : \mathcal{S}^1 \to \mathbb{R}^2$ gives the frontier
- $\mathbf{g} : \mathcal{S}^1 \to \mathbb{R}^2$ gives the normal

Both **analytic expressions** are **unknown** but we can **evaluate the image** of an interval by these functions.

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Fake Boundary

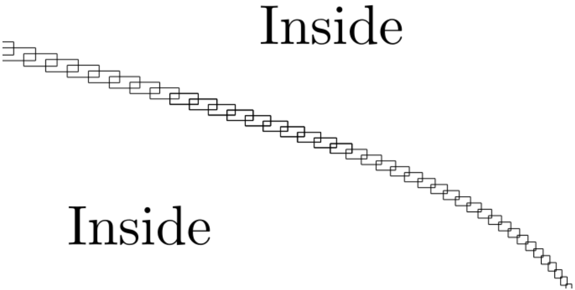

Figure: Fake boundary

Intuition

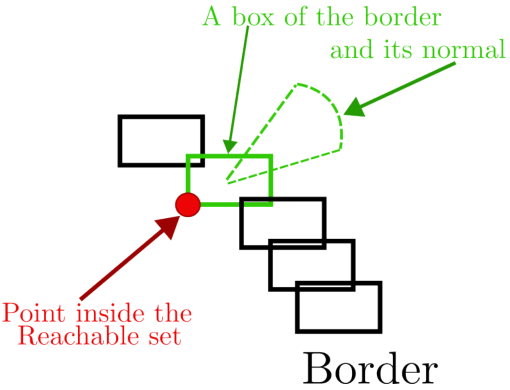

Figure: Proposition 1

Specific case

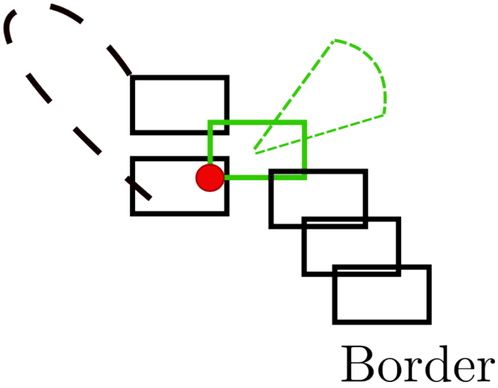

Figure: Case to detect

Neighborhood relation

Definition

Let $[t_i]$ and $[t_j]$ be two real-value intervals. We define the neighborhood relation noted $\mathcal{R}_n$ between $[t_1]$ and $[t_2]$ as :

$$[t_i]\,\mathcal{R}_n\,[t_j] \iff [t_i] \cap [t_j] \neq \emptyset$$

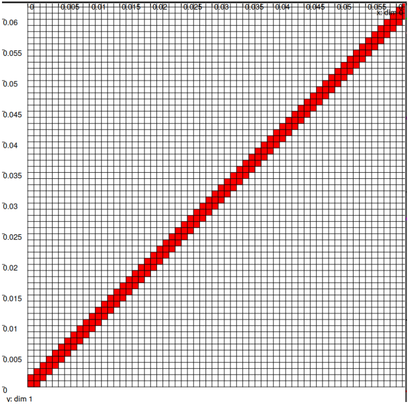

Figure: t-plane representation of the Neighborhood relation

Box Chain relation

Definition

Let there be $[t_i]$ and $[t_k]$ two real-value intervals and $\mathbf{g} : \mathcal{S}^1 \mapsto \mathbb{R}^2$. We define the box chain relation noted $\mathcal{R}_{BC}$ between $[t_i]$ and $[t_k]$ as :

$$[t_i]\,\mathcal{R}_{BC}\,[t_k] \iff \exists\,[t_{j_1}],[t_{j_2}],\ldots,[t_{j_m}] \in \mathbb{R}^n,$$

$$\left([t_i]\,\mathcal{R}_n\,[t_{j_1}] \cap [t_{j_1}]\,\mathcal{R}_n\,[t_{j_2}] \cdots \cap [t_{j_m}]\,\mathcal{R}_n\,[t_k]\right) \cap$$

$$0 \notin [\![\mathbf{g}]([t_i]),[\mathbf{g}]([t_{j_1}]),\ldots,[\mathbf{g}]([t_{j_m}]),[\mathbf{g}]([t_k])]\!]$$

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Box Chain relation

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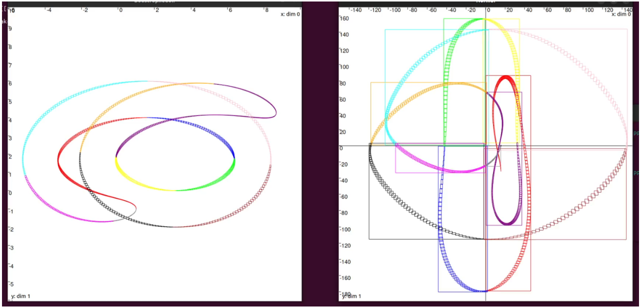

Figure: Show video

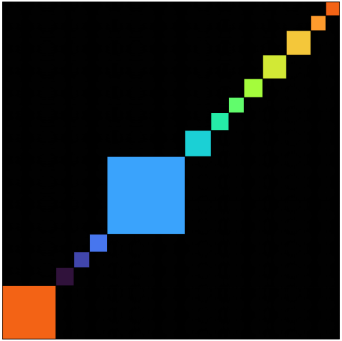

Figure: t-plane representation of the BoxChain Relation

Detecting intersections

Figure: Box Chain decomposition

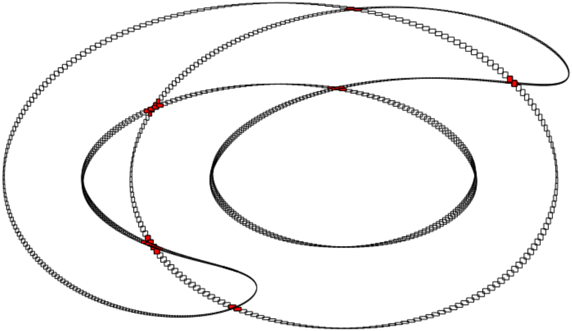

Figure: Intersections detected

Propostion 1

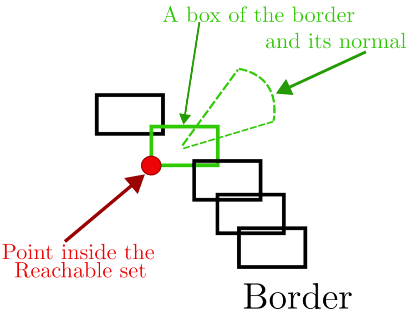

Figure: Propostion 1

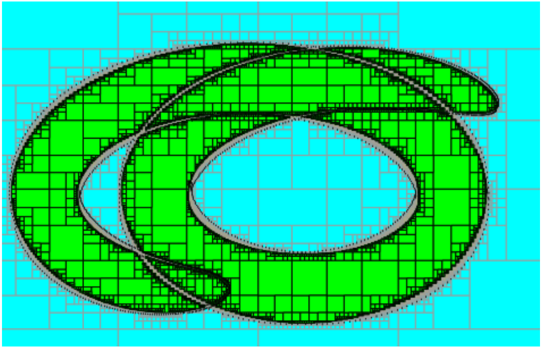

Figure: Interior detection

Propostion 2

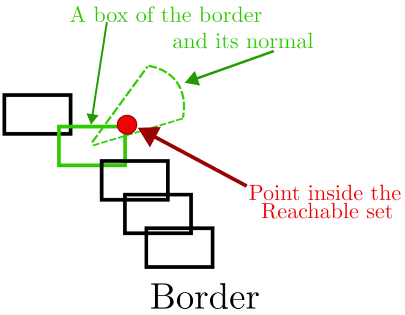

Figure: Proposition 2

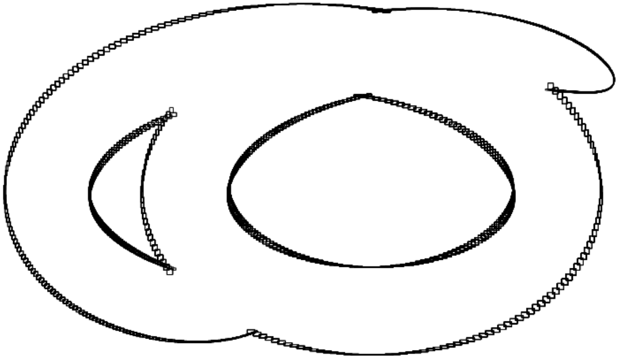

Figure: Fake boundaries deleted

3D Box Chains

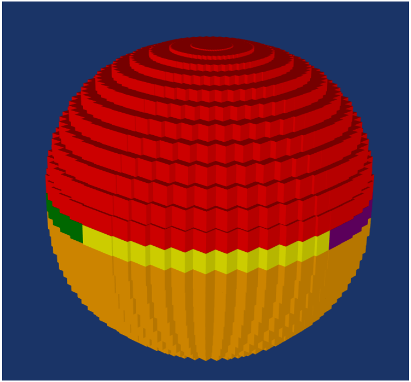

Figure: Show video

Thank you for listening

Bibliography

[1] Lew T., Bonnali R., Pavone M., Exact Characterization of the Convex Hulls of Reach- able Sets, 62nd IEEE Conference on Decision and Control (CDC 2023), Dec 2023, Singapour, Singapore.

Appendix

$$\mathbf{ODE}_{w(0)}: \begin{cases} \dot{x}(t) = f(x(t)) + (n^{\partial \mathcal{W}})^{-1}(q(t)), \\ \dot{q}(t) = -\mathsf{Proj}_{q(t)}\left(\nabla f(x(t))^\top q(t)\right), \\ (x(0), q(0)) = (x^0, n^{\partial \mathcal{W}}(w(0))). \end{cases} t \in [0, T], \tag{7}$$

Figure: ODE