Boundary simplification

Conclusion 000

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# Introducing Box Chains to simplify Reachability Analysis

### $M. \ {\sf Godard}^1 \quad {\sf L}. \ {\sf Jaulin}^1 \quad {\sf D}. \ {\sf Masse}^2$

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Summer Workshop on Interval Methods, 2024





- 2 Reachability analysis
- Boundary simplification





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Boundary simplification

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### Introductive Problem



Figure: Helios



Reachability analysis

Boundary simplification

Conclusion

### Introductive Problem







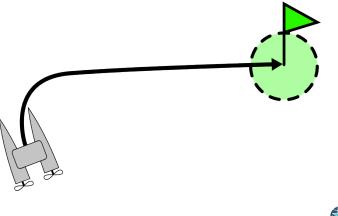
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### Perfect case



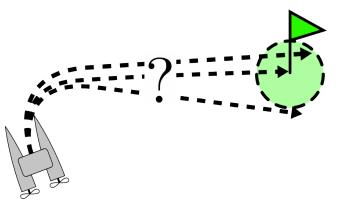


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### Perturbated case





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### Definition

#### Definition (Reachable set at a point in time)

Consider a dynamical system following a state equation of the form  $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ . The set of initial states and inputs are bounded i.e  $\mathbf{x}(0) \in \mathbb{X}_0$  and  $\mathbf{u} \in \mathbb{U}$ . The reachable set at a certain point of time tr is defined as the union of the possible system states at  $t = t_r$ :

$$\mathcal{R}(t_r) = \left\{ \mathbf{z} \in \mathbb{R} | \int_0^{t_r} \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) dt, \mathbf{x}(0) \in \mathbb{X}_0, \mathbf{u}[0, t_r] \in \mathbb{U} \right\}$$

where  $\mathbf{u}[0, t_r] = \bigcup_{t \in [0, t_r]} \mathbf{u}(t)$ 



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### Illustration





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Reachability analysis

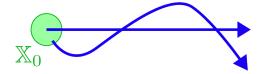
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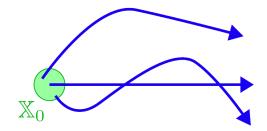




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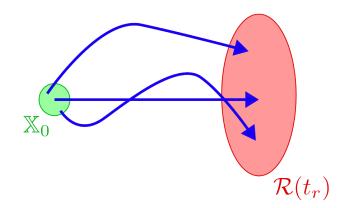




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### Starting result

### From [1] (Thomas $\rm Lew$ 2023) applied with Interval Analysis tools

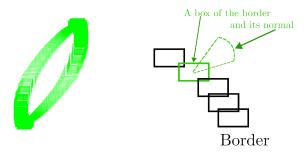


Figure: Frontier of the Reachable Set

Each box is the result of a guaranteed integration



Boundary simplification

Conclusion

### Self-intersecting frontier

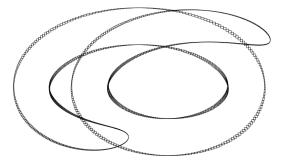
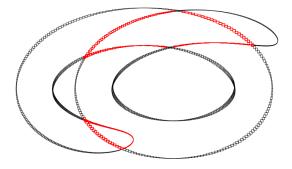


Figure: Self-intersecting frontier



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#### Figure: Fake boundaries



Reachability analysis

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We define two functions :

•  $\mathbf{f}:\mathcal{S}^1 
ightarrow \mathbb{R}^2$  gives the frontier

•  $\mathbf{g}: \mathcal{S}^1 
ightarrow \mathbb{R}^2$  gives the normal

Both analytic expressions are unknown but we can evaluate the image of an interval by these functions.

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Boundary simplification

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# Fake Boundary

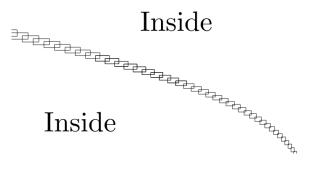


Figure: Fake boundary



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Conclusion

### Intuition

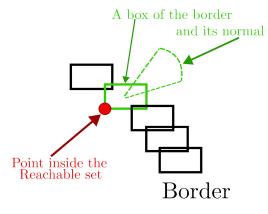


Figure: Proposition 1



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### Specific case

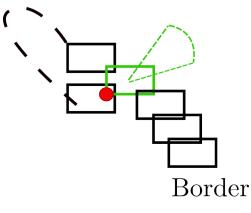


Figure: Case to detect



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# Neighborhood relation

#### Definition

Let  $[t_i]$  and  $[t_j]$  be two real-value intervals. We define the neighborhood relation noted  $\mathcal{R}_n$  between  $[t_1]$  and  $[t_2]$  as :

$$[t_i] \mathcal{R}_n[t_j] \Longleftrightarrow [t_i] \cap [t_j] \neq \emptyset$$



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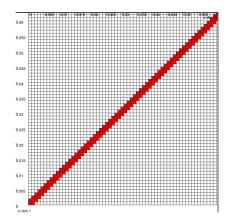


Figure: t-plane representation of the Neighborhood relation



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# Box Chain relation

#### Definition

Let there be  $[t_i]$  and  $[t_k]$  two real-value intervals and  $\mathbf{g} : S^1 \mapsto \mathbb{R}^2$ . We define the box chain relation noted  $\mathcal{R}_{BC}$  between  $[t_i]$  and  $[t_k]$  as :

$$\left[t_{i}
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ight] \Longleftrightarrow \exists \left[t_{j_{1}}
ight]$$
 ,  $\left[t_{j_{2}}
ight]$  ,  $\ldots$  ,  $\left[t_{j_{m}}
ight] \in \mathbb{R}^{n}$  ,

 $([t_i] \mathcal{R}_n[t_{j_1}] \cap [t_{j_1}] \mathcal{R}_n[t_{j_2}] \cdots \cap [t_{j_m}] \mathcal{R}_n[t_k]) \cap$ 

 $0 \notin \llbracket [\mathbf{g}] ([t_i]), [\mathbf{g}] ([t_{j_1}]), \dots, [\mathbf{g}] ([t_{j_m}]), [\mathbf{g}] ([t_k]) \rrbracket$ 



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ight] \in \mathbb{R}^{n}$  ,

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 $0 \notin \llbracket [\mathbf{g}] ([t_i]), [\mathbf{g}] ([t_{j_1}]), \dots, [\mathbf{g}] ([t_{j_m}]), [\mathbf{g}] ([t_k]) \rrbracket$ 



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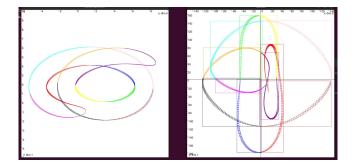
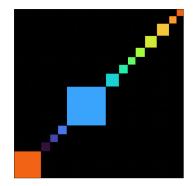


Figure: Show video



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#### Figure: t-plane representation of the BoxChain Relation



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### Detecting intersections

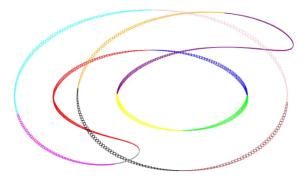


Figure: Box Chain decomposition



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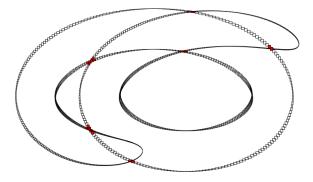


Figure: Intersections detected

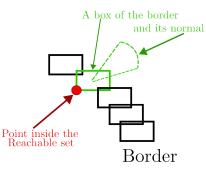


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### Propostion 1



#### Figure: Propostion 1



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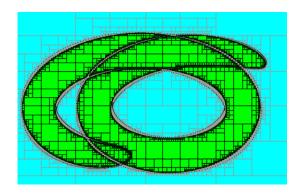


Figure: Interior detection



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### Propostion 2

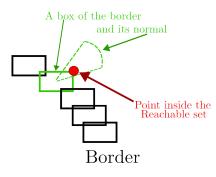


Figure: Proposition 2



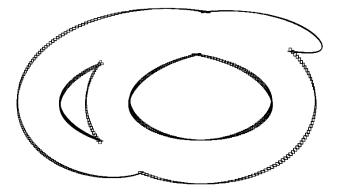


Figure: Fake boundaries deleted



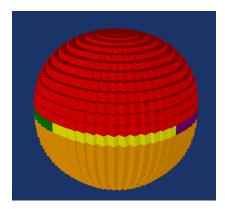
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### 3D Box Chains



#### Figure: Show video



# Thank you for listening



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Boundary simplification

# Bibliography

[1] LEW T., BONNALI *R.*, PAVONE *M.*, Exact Characterization of the Convex Hulls of Reach- able Sets, 62nd IEEE Conference on Decision and Control (CDC 2023), Dec 2023, Singapour, Singapore.



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# Appendix

$$\mathbf{ODE}_{w(0)} \colon \begin{cases} \dot{x}(t) = f(x(t)) + (n^{\partial \mathcal{W}})^{-1} (q(t)) ,\\ \dot{q}(t) = -\mathsf{Proj}_{q(t)} \left( \nabla f(x(t))^{\top} q(t) \right) , \\ (x(0), q(0)) = (x^{0}, n^{\partial \mathcal{W}}(w(0))). \end{cases}$$
(7)

#### Figure: ODE



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