Interval-based validation of a nonlinear estimator

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International Workshop on Reliable Engineering Computing, 2024

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Lab-STICC
COLLEGES BRETAGNE LOIRE SCIENCES POUR L'INGENIEUR ET LE NUMERIQUE
Avec le soutien de
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Introduction

What is an estimator ?

Definition

Estimator In statistics, an estimator is a rule for calculating an estimate of a given quantity based on observed data [Wikipedia]

Example

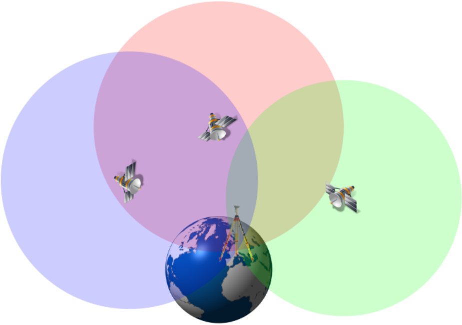

Figure: [1] An estimation : GNSS positioning

Problematic

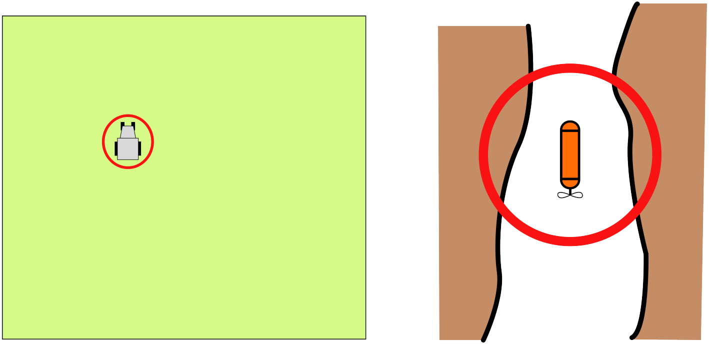

Figure: Error of the estimation

Definitions

$\mathbf{x} \in \mathbb{X}_0$ Set of possible parameters

$\mathbf{y} = \mathbf{g}(\mathbf{x}) + \mathbf{e}$ Noisy observation

$\mathbf{e} \in \mathbb{E}$ Noise set

$\hat{\mathbf{x}} = \psi(\mathbf{y})$ Estimator to validate

$\epsilon(\mathbf{x}) = ||\mathbf{x} - \hat{\mathbf{x}}||$ Error of the estimator

$\bar{\epsilon} = \max(\epsilon(\mathbf{x}))$

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Error from estimation

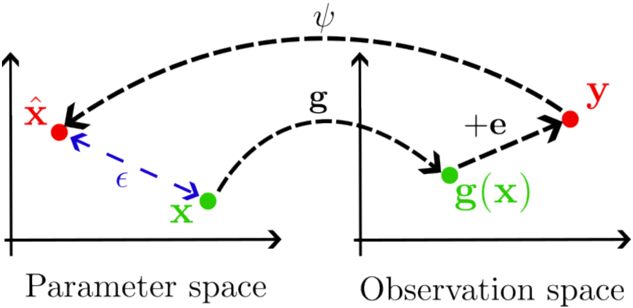

Figure: Error in the estimation

Formalism

Equivalently, this problem can be written as below:

$$\begin{cases} \max \epsilon(\mathbf{x}) = \left\| \mathbf{x} - \psi\left(\mathbf{g}(\mathbf{x}) + \mathbf{e}\right) \right\| \\ \mathbf{x} \in \mathbb{X}_0 \\ \mathbf{e} \in \mathbb{E} \end{cases}$$

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An Optimization problem is defined by:

- An objective function to minimize $f : \mathbb{R}^n \mapsto \mathbb{R}$
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The Moore-Skelboe algorithm

The Moore-Skelboe algorithm gives an box containing the global minimum of a function with width inferior to a choosen criteria, noted δ below:

Let $\{B_0\}$ be a cover of $\mathbb{X}_0$

while $w(f(B_0)) > \delta$ **do** ▷ stopping criterion to choose

Remove B_0 from the cover

Split B_0

Insert the result into the cover in increasing order of $lb(f(B_i))$, for $i = \{0, \ldots, N-1\}$

end while

return $f(B_0)$ ▷ $\mu \in f(B_0)$

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Example

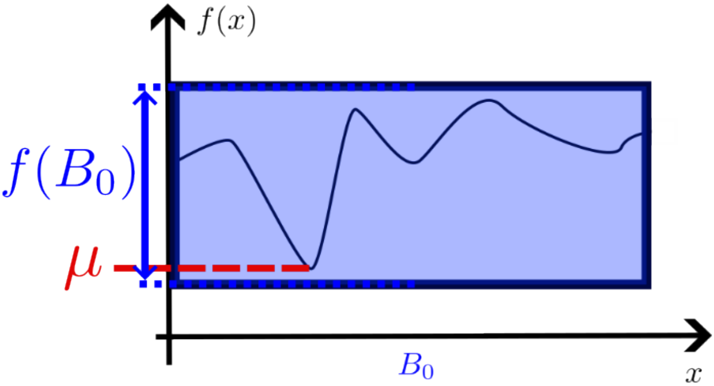


Figure: Moore-Skelboe algorithm - Step 1

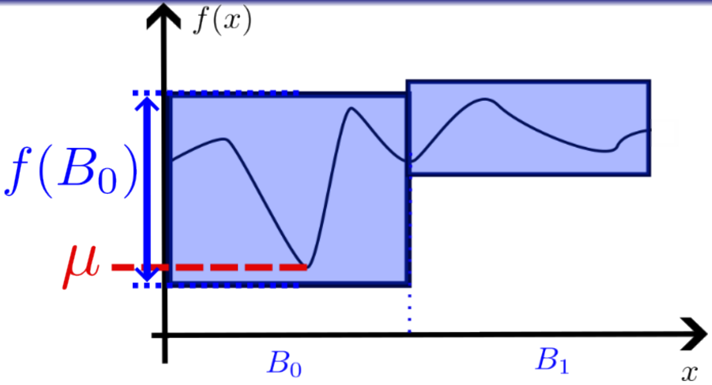


Figure: Moore-Skelboe algorithm - Step 2

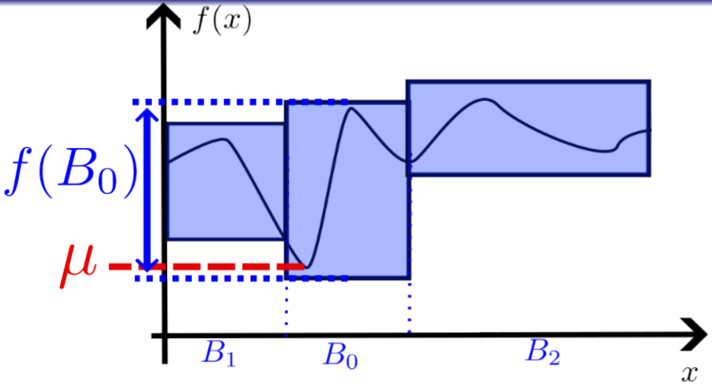


Figure: Moore-Skelboe algorithm - Step 3

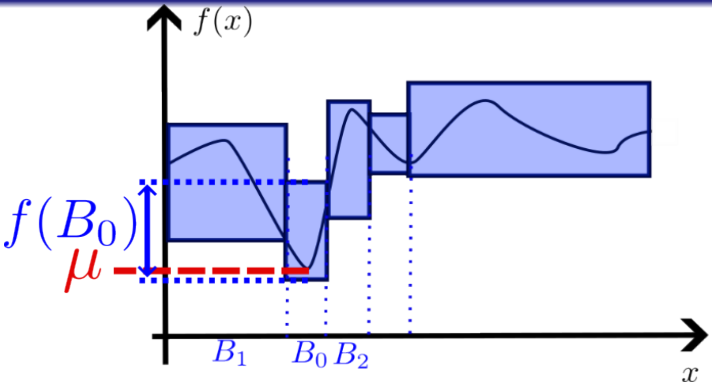
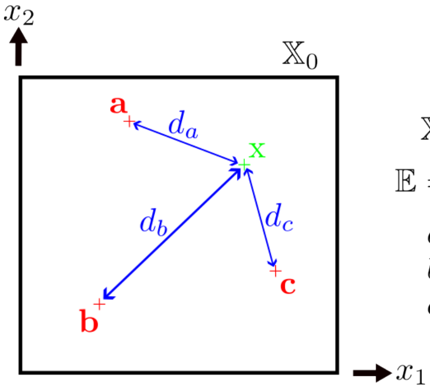


Figure: Moore-Skelboe algorithm - Step 4

Application



$$\mathbb{X}_0 = [5, 25]^2$$
$$\mathbb{E} = [-0.2, 0.2]^3$$
$$a = (10, -9)$$
$$b = (5, 12)$$
$$c = (-15, 0)$$

Figure: Problem description

Problem

Observation Function

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} d_a \\ d_b \\ d_c \end{pmatrix}$$

Error Function

$$\epsilon\left(\mathbf{x}\right) = \left\|\mathbf{x} - \psi\left(\mathbf{g}\left(\mathbf{x}\right) + \mathbf{e}\right)\right\|$$

Moore-Skelboe on $-\epsilon$

CNN Estimator

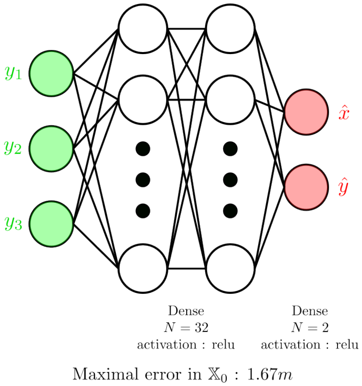


Figure: Neural Network Estimator

Simulation

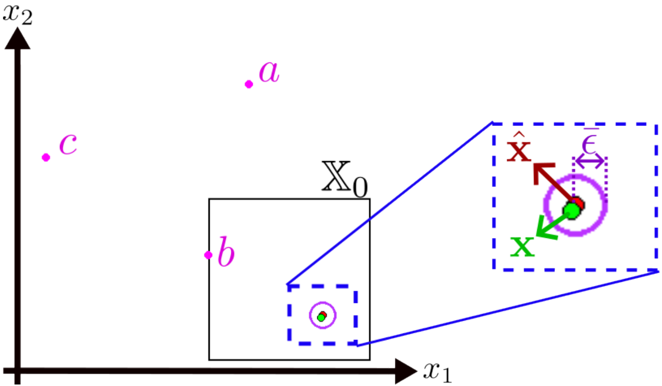


Figure: Visualization of $\bar{\epsilon}$

Conclusion

- Validation of a nonlinear estimator (non interval-based)
- Guaranteed result even without a guaranteed estimator

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Questions

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Bibliography

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[2] VAN EMDEN M., MOA B., Termination Criteria in the Moore-Skelboe Algorithm for Global Optimization by Interval Arithmetic, 2004.

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