Enclosing of the image of a sphere by a nonlinear function using parallelepipeds

Maël Godard[1] Luc Jaulin[1] Damien Massé[2]

[1]Lab-STICC, ROBEX Team, ENSTA

[2]Lab-STICC, ROBEX Team, UBO

International Online Seminar on Interval Methods in Control Engineering

ENSTA IP PARIS
Lab-STICC
COLLEGES SCIENCES BRETAGNE POUR L'INGENIEUR LOIRE ET LE NUMERIQUE
Avec le soutien de
Région Bretagne
AGENCE INNOVATION DÉFENSE

Motivation

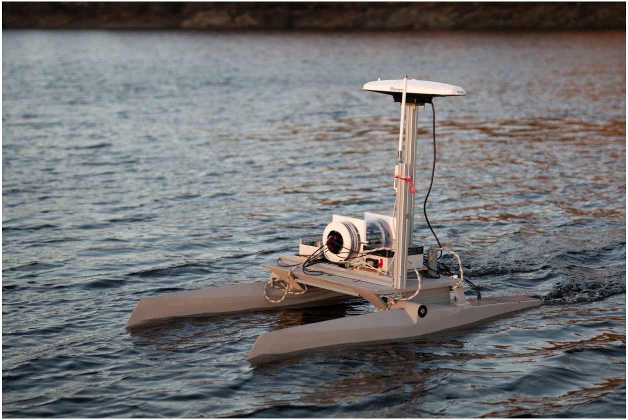

Figure: Autonomous robot Helios

ENSTA INSTITUT POLYTECHNIQUE DE PARIS

Objective

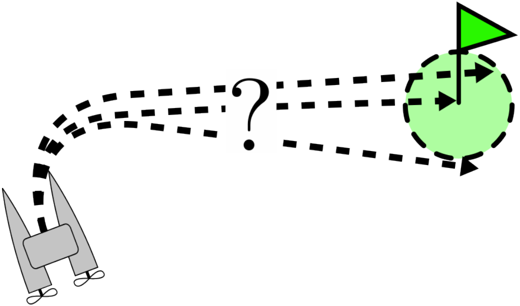

Figure: Thesis' objective

Flow function

Definition (Flow function)

Consider a dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \mathbf{x}(0) \in \mathbb{X}_0, \mathbf{u} \in \mathcal{U}$$

This equation admits a unique solution called flow function, noted $\phi : \mathbb{X}_0 \times \mathcal{U} \times \mathbb{R} \to \mathbb{R}^n$, that satisfies:

$$\forall(\mathbf{x}_0, \mathbf{u}(.), t) \in \mathbb{X}_0 \times \mathcal{U} \times \mathbb{R}, \phi(\mathbf{x}_0, \mathbf{u}(.), t) = \mathbf{x}(t)$$

ENSTA INSTITUT POLYTECHNIQUE DE PARIS

Reachable set

Definition (Reachable set at a point in time)

Consider a dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \mathbf{x}(0) \in \mathbb{X}_0, \mathbf{u} \in \mathcal{U}$$

The reachable set at time t_r noted $\mathcal{R}(t_r)$ can then be defined by :

$$\mathcal{R}(t_r) = \{\mathbf{x} \in \mathbb{R}^n | \exists \mathbf{x}_0 \in \mathbb{X}_0, \exists \mathbf{u}(.) \in \mathcal{U}, \phi(\mathbf{x}_0, \mathbf{u}(.), t_r) = \mathbf{x}\}$$

Reachability Analysis

From the litterature [1], finding the reachable set at time t from an initial state $\mathbf{x}_0$ comes down to integrating the ODE :

$$\mathbf{ODE}_{w(0)} : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + (n^{\partial U})^{-1}(q(t)) \\ \dot{\mathbf{q}}(t) = -Proj_{\mathbf{q}(t)}(\nabla \mathbf{f}(\mathbf{x}(t))^T \mathbf{q}(t) \\ (\mathbf{x}(0), \mathbf{q}(0)) = (\mathbf{x}_0, n^{\partial U}(\mathbf{u}(0))) \end{cases}$$

On the boundary of $\mathbb{X}_0$

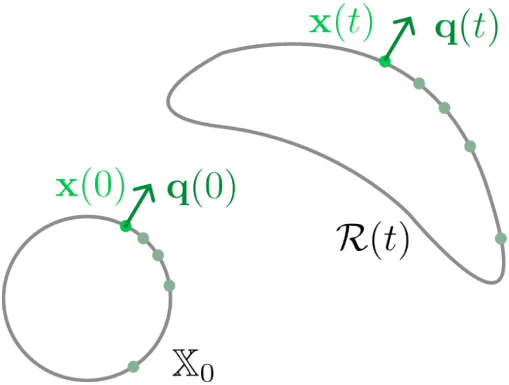

Figure: Reachable set

Wrapping effect

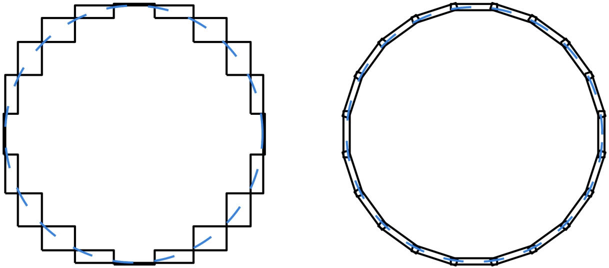

Figure: Enclosing of the unit circle with boxes and parallelepipeds

Notations and definitions

- The studied function is smooth, at least C^1.
- We denote $\mathcal{S}^n$ the unit sphere of dimension n
- We limit our ODEs to the ones of the form:

$$\dot{\mathbf{x}} = \gamma(\mathbf{x})$$

Parallelepiped definition

Definition (Parallelepiped)

A *parallelepiped* is a subset of $\mathbb{R}^n$ of the form

$$\langle \mathbf{y} \rangle = \bar{\mathbf{y}} + \mathbf{A} \cdot [-1, 1]^m = \{\bar{\mathbf{y}} + \mathbf{A} \cdot \mathbf{x} \mid \mathbf{x} \in [-1, 1]^m\}$$

With $m \leq n$

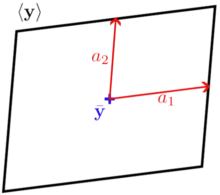

Figure: 2D parallelepiped

Parallelepiped inclusion function

Definition (Parallelepiped inclusion function)

A *parallelepipedic inclusion function* is a function

$$\langle \mathbf{f} \rangle : \begin{array}{ccc} \mathbb{IR}^m & \rightarrow & \mathbb{PR}^n \\ [\mathbf{x}] & \rightarrow & \langle \mathbf{f} \rangle([\mathbf{x}]) \end{array}$$

such that

$$\mathbf{f}([\mathbf{x}]) \subset \langle \mathbf{f} \rangle([\mathbf{x}])$$

And $\langle \mathbf{f} \rangle([\mathbf{x}])$ is a parallelepiped.

ENSTA INSTITUT POLYTECHNIQUE DE PARIS

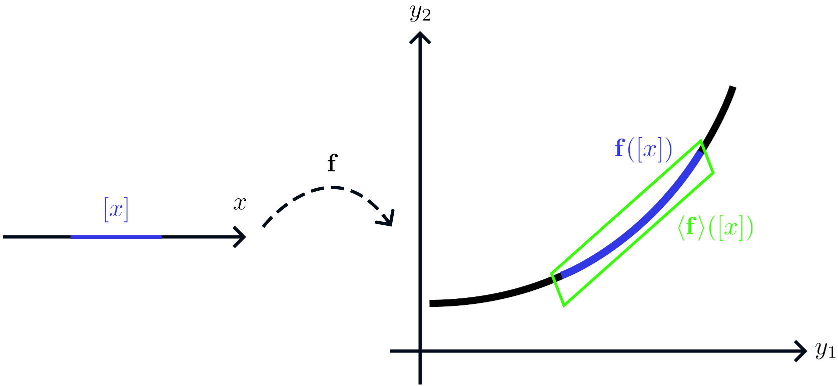

Figure: Parallelepiped inclusion function

Approximation theorem

Theorem

Consider a smooth function $\mathbf{f}$ *from* $\mathbb{R}^m$ *to* $\mathbb{R}^n$*, and a box* $[\mathbf{x}] \in \mathbb{IR}^m$ *with center* $\bar{\mathbf{x}}$ *. Define the linear approximation*

$$\ell(\mathbf{x}) = \mathbf{f}(\bar{\mathbf{x}}) + \frac{d\mathbf{f}}{d\mathbf{x}}(\bar{\mathbf{x}}) \cdot (\mathbf{x} - \bar{\mathbf{x}})$$

We then have

$$\forall \mathbf{x} \in [\mathbf{x}], \|\mathbf{f}(\mathbf{x}) - \ell(\mathbf{x})\| \leq \rho$$

where

$$\rho = \rho_{\mathbf{f}}([\mathbf{x}]) = ub\left(\left\|\left(\left[\frac{d\mathbf{f}}{d\mathbf{x}}\right]([\mathbf{x}]) - \frac{d\mathbf{f}}{d\mathbf{x}}(\bar{\mathbf{x}})\right) \cdot ([\mathbf{x}] - \bar{\mathbf{x}})\right\|\right)$$

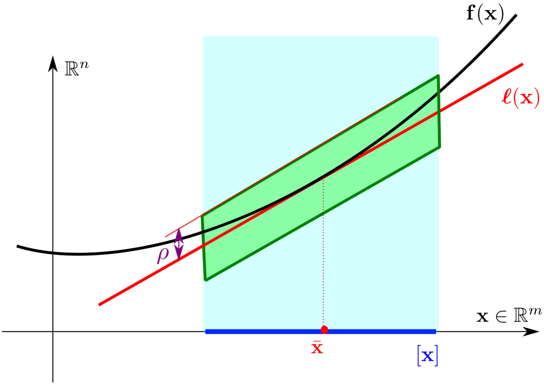

Figure: Approximation theorem

Corollary

Corollary

Given a function $\mathbf{f}$ *from* $\mathbb{R}^m$ *to* $\mathbb{R}^n$, *and a box* $[\mathbf{x}] \in \mathbb{IR}^m$. *We have*

$$\mathbf{f}([\mathbf{x}]) \subset \ell([\mathbf{x}]) + \rho\mathbb{U}$$

where $\mathbb{U}$ *is the unit sphere,* $\rho = \rho_{\mathbf{f}}([\mathbf{x}])$ *and* $\ell([\mathbf{x}])$ *is the linear approximation defined earlier.*

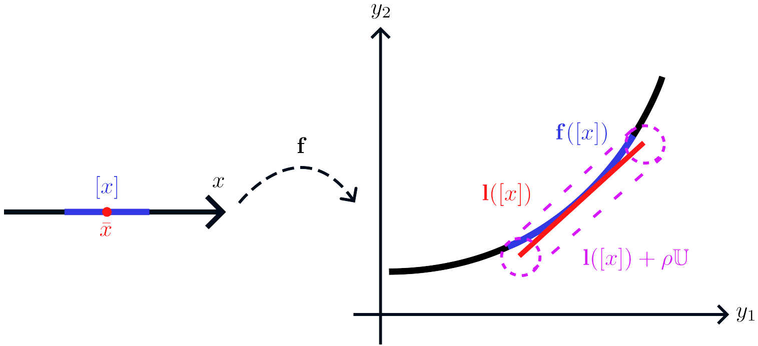

Figure: Corollary in 2D

Parallelepiped inflation

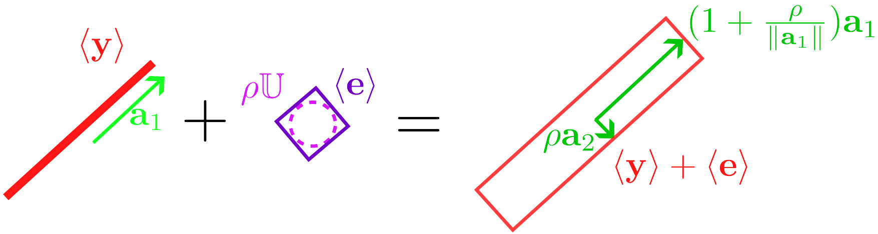

Figure: Parallelepiped inflation in 2D

Parallelepipedic inclusion function

Given $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$. A parallelepipedic inclusion function is obtained as follows:

$$\langle \mathbf{f} \rangle([\mathbf{x}]) = \bar{\mathbf{y}} + \mathbf{A} \cdot [-1, 1]^n$$

with

$$\begin{array}{rcc} \bar{\mathbf{y}} & = & \mathbf{f}(\bar{\mathbf{x}}) \\ \mathbf{A}_0 & = & \frac{d\mathbf{f}}{d\mathbf{x}}(\bar{\mathbf{x}}) \cdot rad([\mathbf{x}]) \\ \rho & = & \mathrm{ub}\left(\left\| \left(\left[\frac{d\mathbf{f}}{d\mathbf{x}} \right] ([\mathbf{x}]) - \frac{d\mathbf{f}}{d\mathbf{x}}(\bar{\mathbf{x}}) \right) \cdot ([\mathbf{x}] - \bar{\mathbf{x}}) \right\|\right) \\ \mathbf{A} & = & \mathrm{Inflate}(\mathbf{A}_0, \rho) \end{array}$$

Box Atlas

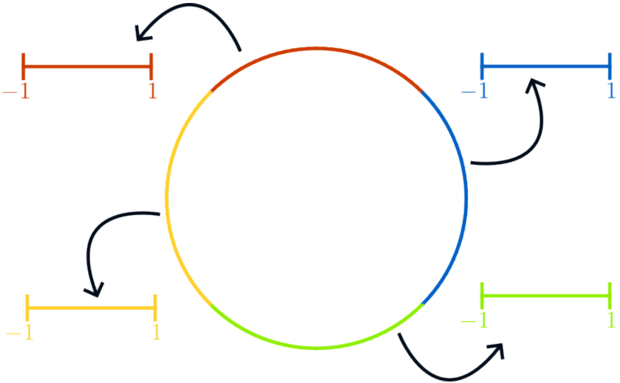

Figure: Box atlas [2]

Image of an interval by a nonlinear function

Assume we want to compute the image of $[x] = [-1, 1]$ by the function ψ_0 defined by

$$\forall x \in [x], \psi_0(x) = \begin{pmatrix} sin(\frac{\pi x}{4}) \\ cos(\frac{\pi x}{4}) \end{pmatrix}$$

The Jacobian matrix is

$$\frac{d\psi_0}{dx} = \frac{\pi}{4} \begin{pmatrix} cos(\frac{\pi x}{4}) \\ -sin(\frac{\pi x}{4}) \end{pmatrix}$$

ENSTA INSTITUT POLYTECHNIQUE DE PARIS

Subdivisions	ϵ	ρ
1	2	0.60
4	0.5	0.042
10	0.2	6.4e-3
20	0.1	1.6e-3

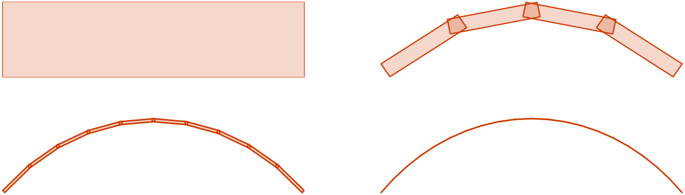

Figure: Approximations of $\psi_0([-1, 1])$

Parallelepipedic inclusion of the circle

Let us define $s_1 = e^{\mathbf{k}\frac{\pi}{2}}$ the rotation of $\frac{\pi}{2}$ with respect to **k**. The parallelepipedic inclusion of the circle can be obtained by the symmetries :

$$\Sigma = \{1, s_1, s_1^2, s_1^{-1}\}$$

The unit circle then corresponds to:

$$\mathcal{S}^1 = \bigcup_{\sigma \in \Sigma} \sigma \circ \psi_0([-1, 1])$$

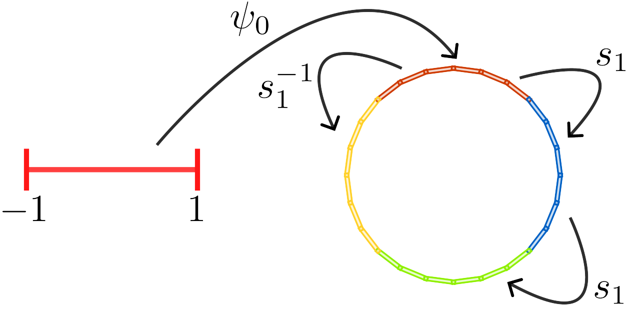

Figure: Parallelepipedic inclusion of the unit circle $\mathcal{S}^1$

Image of the unit circle by a nonlinear function

Consider a function $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. We can then write the image of the unit circle $\mathcal{S}^1$ by $\mathbf{f}$ as :

$$\mathbf{f}(\mathcal{S}^1) = \bigcup_i \langle \mathbf{g}_i \rangle ([-1, 1])$$

Where $\langle \mathbf{g}_i \rangle$ is a parallelepiped inclusion function of :

$$\mathbf{g}_i = \mathbf{f} \circ \sigma_i \circ \psi_0$$

For graphical purposes, we consider the Henon map defined by :

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_2 + 1 - a x_1^2 \\ b x_1 \end{pmatrix}, a = 1.4, b = 0.3$$

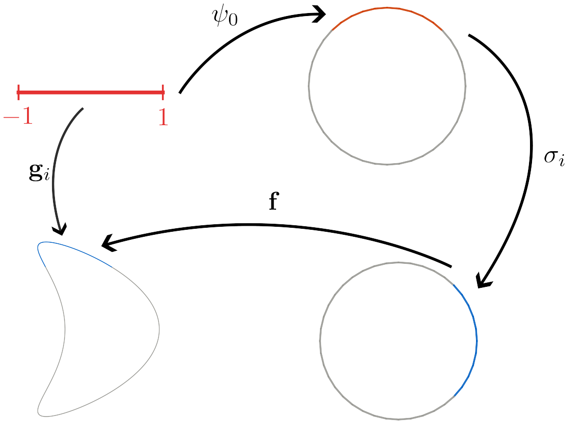

Figure: Image of the unit circle by the Henon map

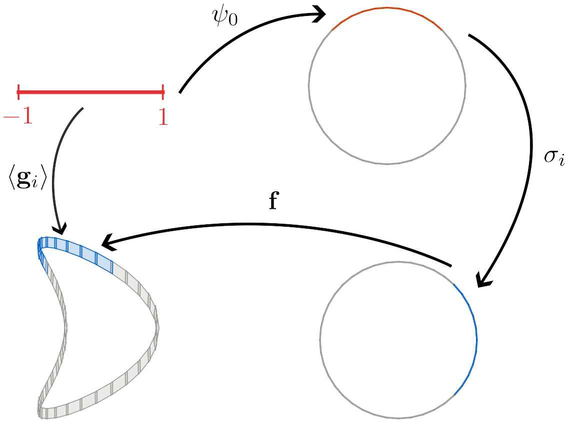

Figure: Image of the unit circle by the Henon map

Comparison

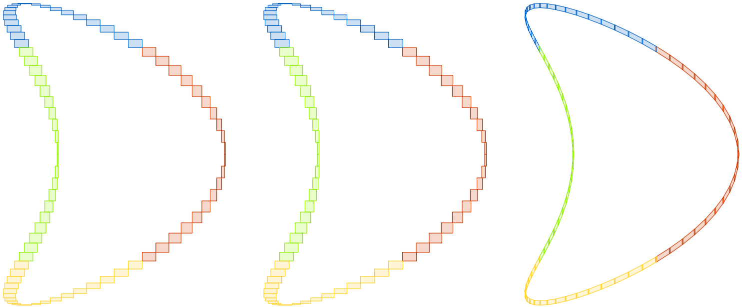

Figure: Natural, Centered [3] [4] and Parallelepipedic inclusion for 20 subdivisions

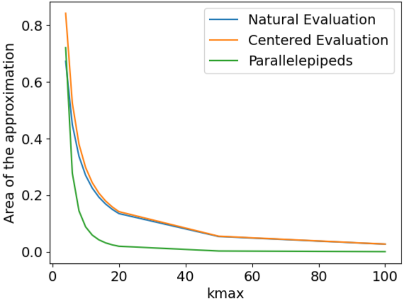

Figure: Area of the approximation depending on the number of subdivisions

Convergence of the parallelepiped inclusion

Let us denote by A the area of the approximation and ϵ the width of a subdivision. If we note k the number of subdivsions of $[-1, 1]$, $\epsilon = \frac{2}{k}$.
If A converges in ϵ^n then

$$\frac{A}{\epsilon^n} \xrightarrow[\epsilon \to 0]{} c, c \in \mathbb{R}$$

Then

$$log(A) \xrightarrow[\epsilon \to 0]{} log(c) + n \cdot log(\epsilon)$$

ENSTA INSTITUT POLYTECHNIQUE DE PARIS

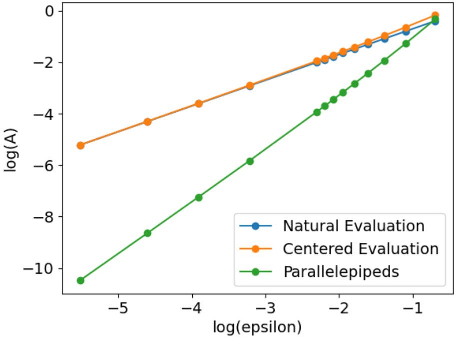

Figure: Convergence of the natural, centered and parallelepedic inclusion

The parallelepiped inclusion seems to **converge in** ϵ^2

Variationnal equation

Consider the system

$$\dot{\mathbf{x}} = \gamma(\mathbf{x})$$
$$\mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{S}^1$$

The solution of this ODE is the flow function $\phi_{\mathbf{x}_0}(t)$. If we denote $\mathbf{A}(\mathbf{x}_0, t) = \frac{\partial \mathbf{\Phi}_{\mathbf{x}_0}}{\partial \mathbf{x}_0}(t)$. We have the *variational equation*

$$\dot{\mathbf{A}} = \frac{d\gamma}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{A}$$

Image of the unit circle by an ODE

Integrating the ODE :

$$\begin{cases} \dot{\mathbf{x}} = \gamma(\mathbf{x}) \\ \dot{\mathbf{A}} = \frac{\partial \gamma}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{A} \\ \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{S}^1 \end{cases}$$

over a time t for any $\mathbf{x}_0 \in \mathcal{S}^1$ will output both $\phi_{\mathbf{x}_0}(t)$ and $\frac{\partial \mathbf{\Phi}_{\mathbf{x}_0}}{\partial \mathbf{x}}(t)$.

Integration of the pendulum with CAPD

Consider the equation of the pendulum:

$$\dot{\mathbf{x}} = \gamma \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -5 \cdot sin(x_1 - 0.5) - 0.5x_2 \end{pmatrix}$$

We then have :

$$\frac{d\gamma}{d\mathbf{x}}(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ -5 \cdot cos(x_1 - 0.5) & -0.5 \end{pmatrix}$$

We integrate the ODE with CAPD [5] and use the parallelepipedic inclusion.

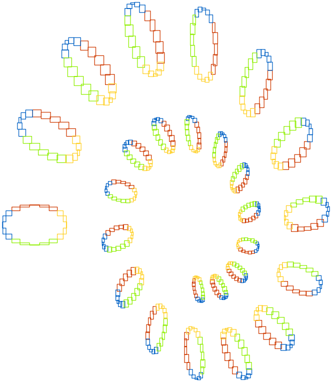

Figure: Integration of the pendulum for 5sec with CAPD alone for 5 subdivisions

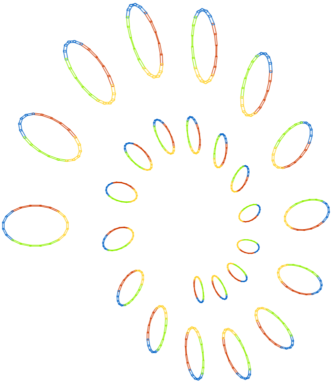

Figure: Integration of the pendulum for 5sec with CAPD an parallelepipeds for 5 subdivisions

Comparison

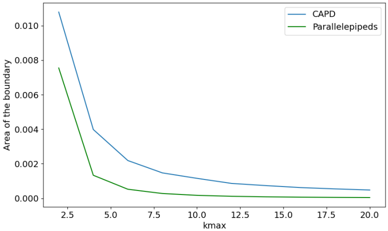

Figure: Area of the boundary after 5sec of integration

3D parallelepiped inflation

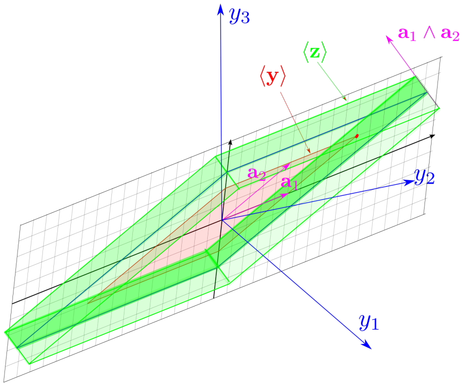

Figure: 3D parallelepiped inflation

Lorenz system

The Lorenz system is defined by :

$$\dot{\mathbf{x}} = \gamma \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \sigma(x_2 - x_1) \\ \rho x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - \beta x_3 \end{pmatrix}, \sigma = 10, \rho = 28, \beta = \frac{8}{3}$$

We then have

$$\frac{d\gamma}{d\mathbf{x}}(\mathbf{x}) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & -x_1 \\ x_2 & x_1 & -\beta \end{pmatrix}$$

We integrate the ODE with CAPD and use the parallelepipedic inclusion.

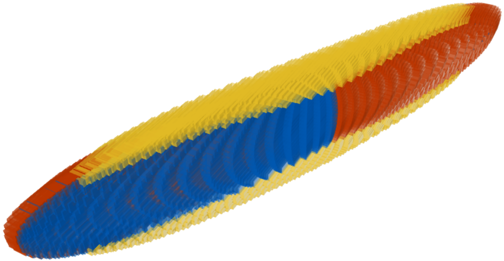

Figure: Lorenz system after 0.1s of integration with capd alone ($\epsilon = 0.1$)

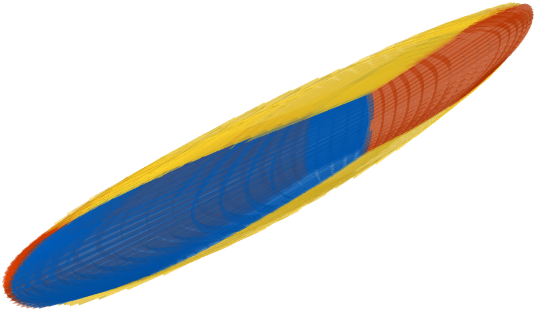

Figure: Lorenz system after 0.1s of integration with CAPD and parallelepiped inclusion ($\epsilon = 0.1$)

Preservation of the topology

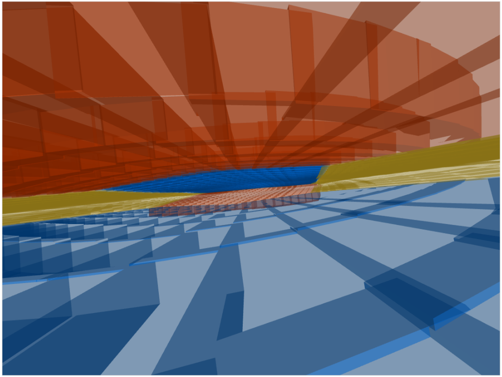

Figure: Inside of the image set

Comparison

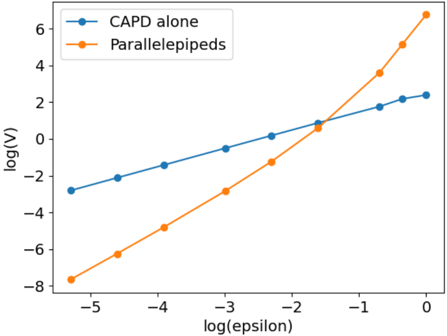

Figure: Convergence of both methods

The parallelepiped inclusion seems to **converge in** ϵ^2

Conclusion

- Parallelepipedic inclusion working in $\mathbb{R}^n$
- Seems to converge in ϵ^2, to prove
- Studying the ODE in the form $\dot{\mathbf{x}} = \gamma(\mathbf{x}, \mathbf{u})$ for Reachability Analysis
- Projection of the enclosure

Thank you for listening

Bibliography

[1] Lew *T.*, Bonnali *R.*, Pavone *M.*, Exact Characterization of the Convex Hulls of Reachable Sets, 62nd IEEE Conference on Decision and Control (CDC 2023), Dec 2023, Singapour, Singapore.

[2] Ignazi A., Manifold decomposition for interval analysis, SWIM 2023

[3] Moore *R.E.*, Methods and Applications of Interval Analysis, 1979, SIAM, Philadelphia.

[4] Rohou S., Desrochers B., Le Bars F., The Codac Library: A Catalog of Domains and Contractors, Acta Cybernetica - Special Issue of SWIM 2022, 2024.

[5] Kapela R., Mrozek M., Wilczak D., Zgliczynski P., CAPD::DynSys: A flexible C++ toolbox for rigorous numerical analysis of dynamical systems, Communications in Nonlinear Science and Numerical Simulation, 2021.