









PhD defense

Safe navigation of a robot in a landmarked environment

Maël Godard, Luc Jaulin (director), Damien Massé (supervisor)

October 13, 2026

Introduction

Inner and outer approximation of the image of a set by a nonlinear function

Adaptive parallelepiped enclosure

Inner and outer approximations of the reachable set under disturbances

Fake boundary removal

Conclusion

Plan

Introduction

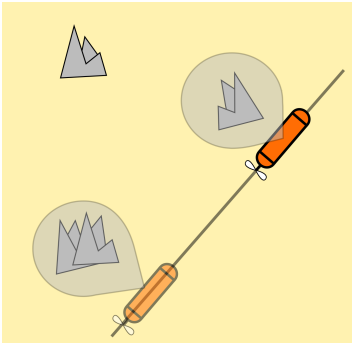
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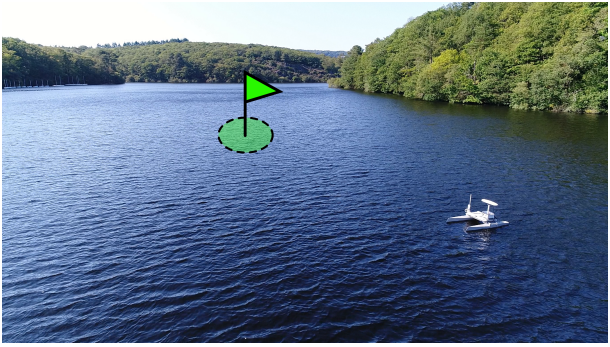
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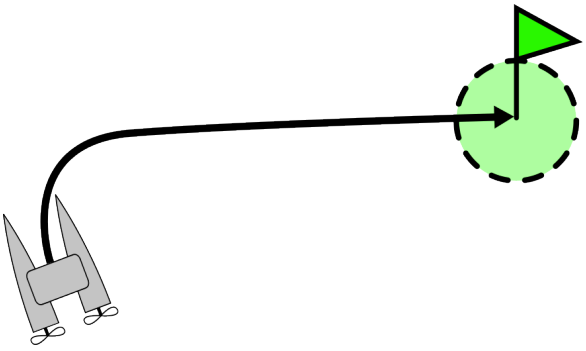
Using landmarks for navigation

Experimental validation



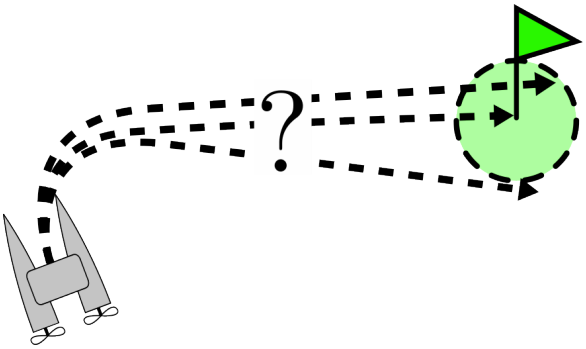
Thesis objective : Will the robot reach the desired area?

Trajectory of a robot: Perfect case



Perfect case

Trajectory of a robot: Disturbed case



Disturbed case

Definition

Consider the dynamical system

$$\dot{\mathbf{x}}(t) = \gamma(\mathbf{x}(t), \mathbf{w}(t)) \tag{1}$$

The reachable set at time t is

$$\mathcal{R}_t(\mathbb{X}_0, \mathbb{W}) = \{\mathbf{x}(t, \mathbf{x}_0, \mathbf{w}) \mid \mathbf{x}_0 \in \mathbb{X}_0,\ \mathbf{w}(\cdot) \in \mathbb{W}\} \tag{2}$$

Since $\mathbf{w}(\cdot)$ can move arbitrarily in $\mathbb{W}$, we treat a *differential inclusion*.

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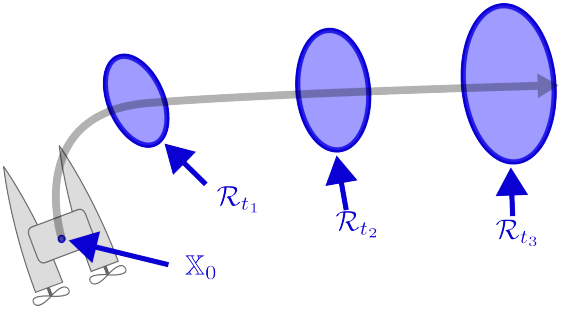
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Reachable set: Illustration



Reachable sets through time

Objectives

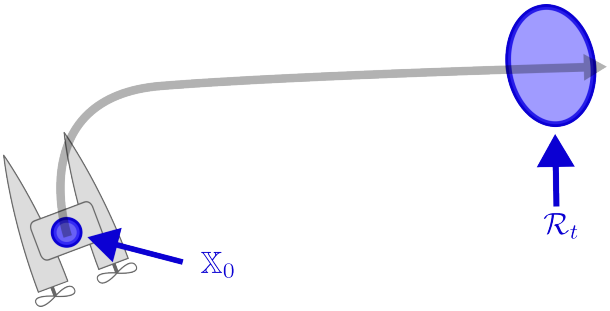
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- of a disturbed dynamical system
- in a guaranteed way

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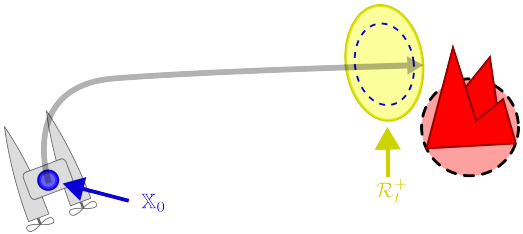


Reachable set to approximate

An *outer approximation* of the reachable set is a superset

$$\mathcal{R}_t^+ \supseteq \mathcal{R}_t \tag{3}$$

"My robot can not end up outside this set"

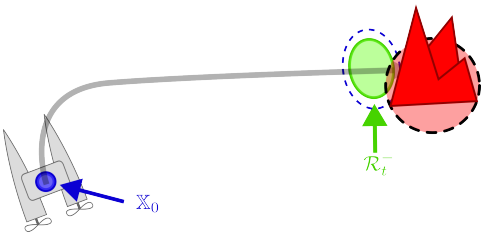


Obstacle is avoided

An *inner approximation* of the reachable set is a subset

$$\mathcal{R}_t^- \subseteq \mathcal{R}_t \tag{4}$$

"My robot can for sure end up in this set"



Risk of collision

Disturbances

A robot is per definition a disturbed system

- Actuators are not perfect
- Subject to external disturbances (wind, current...)
- Noisy sensors affect the navigation

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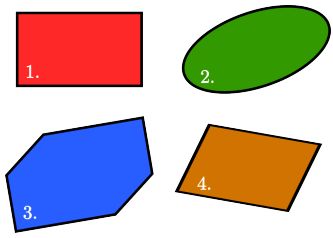
A robot is per definition a disturbed system

- Actuators are not perfect
- Subject to external disturbances (wind, current...)
- Noisy sensors affect the navigation

Under the hypothesis of **bounded uncertainties**, set based methods propagate them in a guaranteed way.

Usual sets are:

1. Interval and boxes (interval vectors)
2. Ellipsoids
3. Zonotopes
4. Parallelepipeds

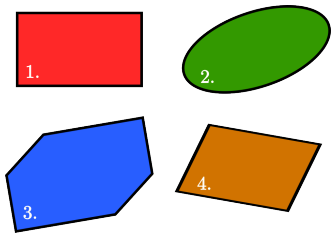


We will use the **CODAC** library for interval arithmetics.

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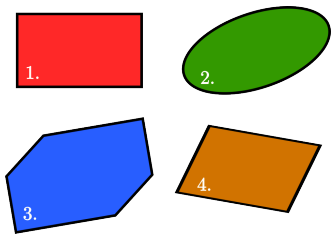


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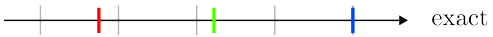
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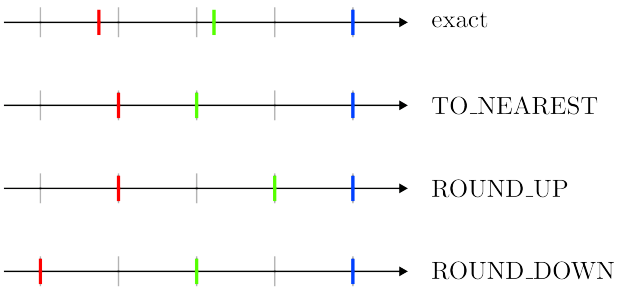


Rounding strategies



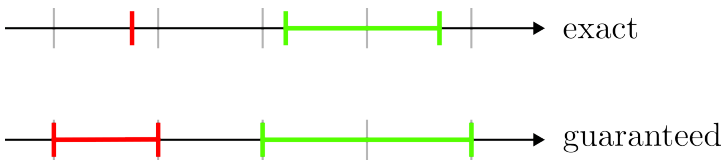
Rounding strategies

Numerical guarantee: Rounding strategies



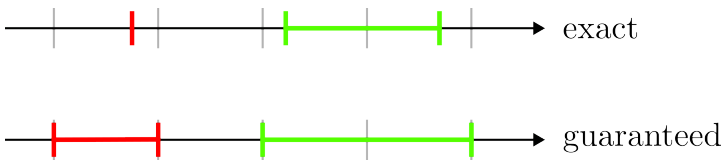
Rounding strategies

Numerical guarantee: Guaranteed implementations



Guaranteed enclosures

CODAC builds upon **GAOL** (pronounced *jail*) which implements this numerical guarantee.



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Thesis positioning

- The inner approximation of the reachable set is less studied than the outer one,
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Objectives

- Inner and outer approximations of the reachable set,
- of a disturbed dynamical system,
- in a guaranteed way.

Let us consider a nonlinear dynamical system. It follows an ODE of the form

$$\dot{\mathbf{x}}(t) = \gamma(\mathbf{x}(t)) \tag{5}$$

If we consider an initial state $\mathbf{x}_0$, we get an Initial Value Problem (IVP)

$$\begin{cases} \dot{\mathbf{x}}(t) = \gamma(\mathbf{x}(t)) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{6}$$

Definition

Consider an IVP. If γ is continuous and C^1, it has a unique solution called the *flow function*. It satisfies

$$\boldsymbol{\phi}(t, \mathbf{x}_0) = \boldsymbol{\phi}_t(\mathbf{x}_0) = \mathbf{x}(t) \tag{7}$$

Definition

Consider an ODE and an initial set $\mathbb{X}_0$. The reachable set of the system at a time t is

$$\mathcal{R}_t(\mathbb{X}_0) = \{\boldsymbol{\phi}_t(\mathbf{x}_0) \mid \mathbf{x}_0 \in \mathbb{X}_0\} = \boldsymbol{\phi}_t(\mathbb{X}_0) \tag{8}$$

Introduction

Inner and outer approximation of the image of a set by a nonlinear function

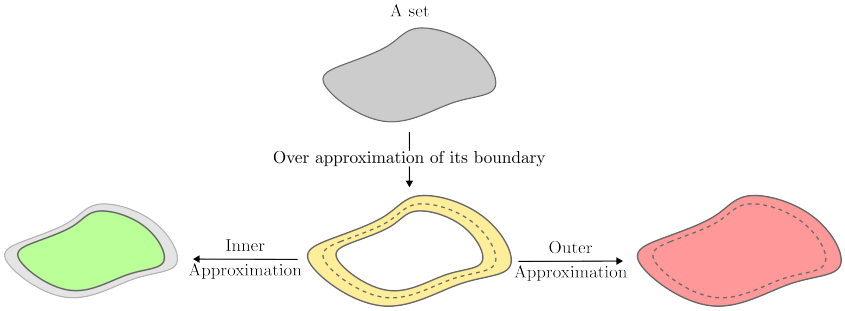
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Our strategy



Boundary based approach

Advantages and disadvantages

+ Lighter in terms of computation

+ Less pessimistic result

- Need a way to “extract” the inner and outer approximations

- Subject to fake boundaries

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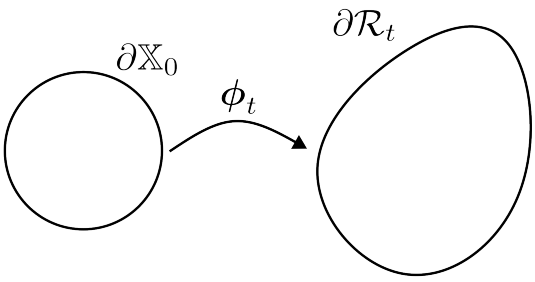
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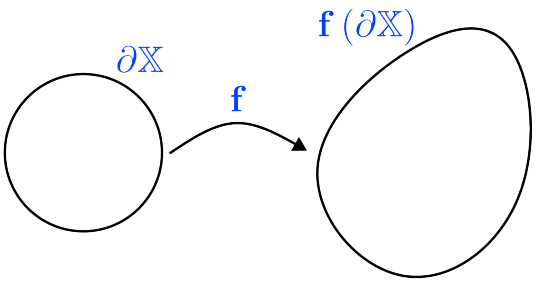
Let us denote $\partial\mathbb{X}_0$ the boundary of $\mathbb{X}_0$. Since $\boldsymbol{\phi}_t$ is bijective and continuous, we have

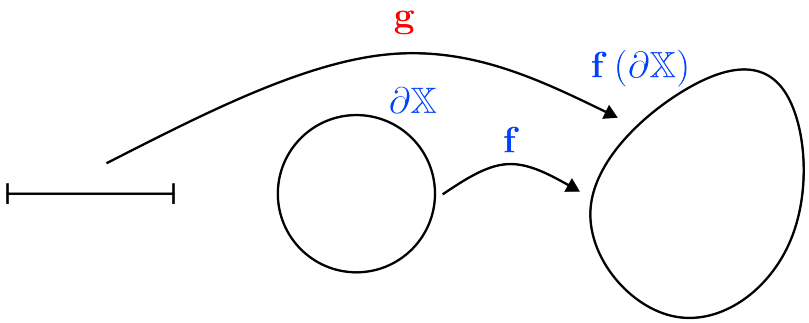
$$\partial\boldsymbol{\phi}_t\left(\mathbb{X}_0\right) = \boldsymbol{\phi}_t\left(\partial\mathbb{X}_0\right) \tag{9}$$

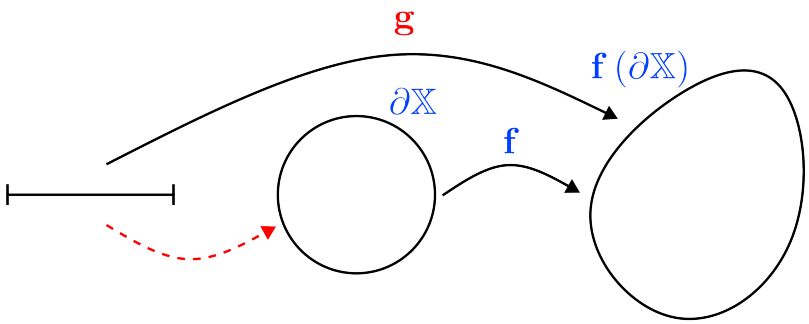
i.e. computing $\boldsymbol{\phi}_t\left(\partial\mathbb{X}_0\right)$ is sufficient.


$\partial\mathcal{R}_t$
$\partial\mathbb{X}_0$
ϕ_t

Objectives




g
f (∂𝕏)
∂𝕏
f


g
f (∂𝕏)
∂𝕏
f

Objectives summary

- Define an inclusion function (for $\mathbf{g}$),
- Use it to compute the image of the boundary of a set,
- Use it for reachability analysis.

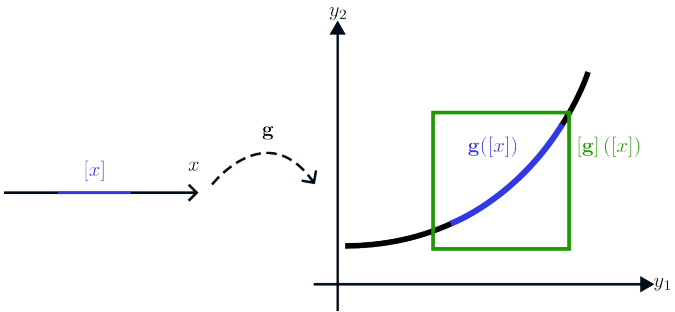
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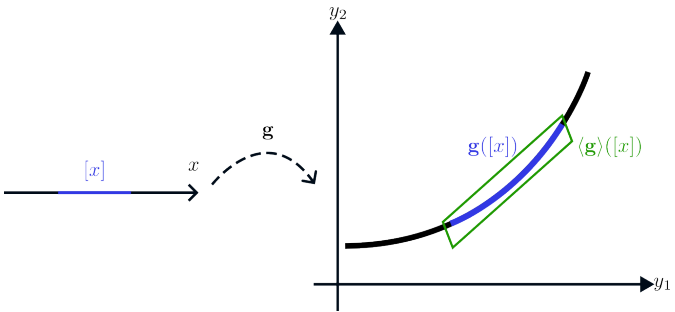
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Consider a function $\mathbf{g} : \mathbb{R} \to \mathbb{R}^2$.

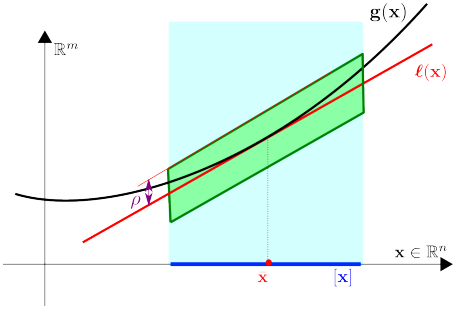


Interval inclusion function

Consider a function $\mathbf{g} : \mathbb{R} \to \mathbb{R}^2$.



Parallelepiped inclusion function



Linear approximation inflation

Consider a box $[\mathbf{x}] \in \mathbb{IR}^n$. Define

$$\ell(\mathbf{x}) = \mathbf{b} + \mathbf{A}(\mathbf{x} - \bar{\mathbf{x}}) \tag{10}$$

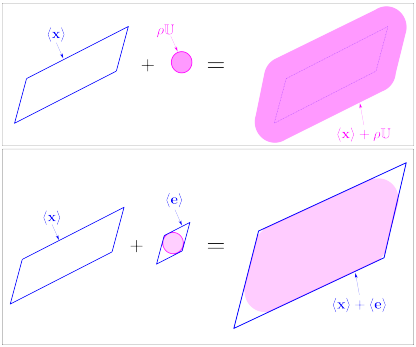
We have

$$\forall \mathbf{x} \in [\mathbf{x}], \|\mathbf{g}(\mathbf{x}) - \ell(\mathbf{x})\| \leq \rho \tag{11}$$

We have for any $\mathbf{x}_c \in [\mathbf{x}]$

$$\rho = \rho_{\mathbf{g}}([\mathbf{x}]) = \text{ub}\left(\left\|[\mathbf{g}](\mathbf{x}_c) - \mathbf{b} + \left(\left[\frac{d\mathbf{g}}{d\mathbf{x}}\right]([\mathbf{x}]) - \mathbf{A}\right) \cdot ([\mathbf{x}] - \mathbf{x}_c)\right\|\right) \tag{12}$$

Parallelepiped inflation[1]



Parallelepiped inflation

[1] **M. Godard**, L. Jaulin, D. Massé, "Inner and outer approximation of the image of a set by a nonlinear function", *International Journal of Approximate Reasoning*, 2025.

To improve the quality of the inclusion, we can introduce a size $\varepsilon \in \mathbb{R}$.

- Split $[\mathbf{x}]$ in subboxes $[\mathbf{x}_s]$ of diameter inferior to ε,
- Compute $\langle \mathbf{g} \rangle([\mathbf{x}_s])$ for each subbox.

Finally,

$$\mathbf{g}([\mathbf{x}]) \subset \bigcup_{[\mathbf{x}_s]} \langle \mathbf{g} \rangle([\mathbf{x}_s]) \tag{13}$$

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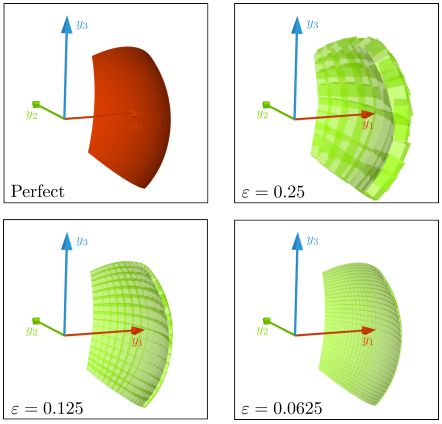
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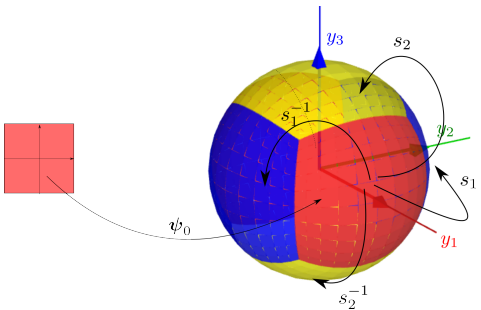
Let us compute the image of $[-1,1]^2$ by

$$\boldsymbol{\psi}_0(\mathbf{x}) = \begin{pmatrix} \frac{1}{\sqrt{1+x_1^2+x_2^2}} \\ \frac{x_1}{\sqrt{1+x_1^2+x_2^2}} \\ \frac{x_2}{\sqrt{1+x_1^2+x_2^2}} \end{pmatrix} \tag{14}$$



Aproximations of $\boldsymbol{\psi}_0\left([-1,1]^2\right)$

- Define a an inclusion function,
- Use it to compute the image of the boundary of a set,
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Gnomonic atlas of the unit sphere

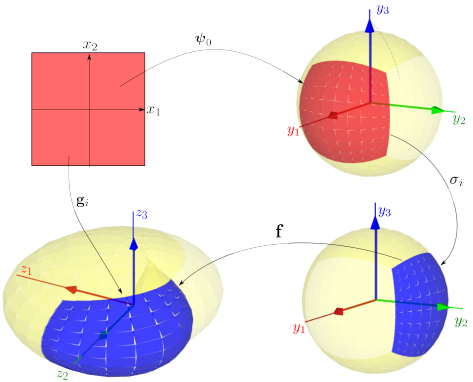
Definition

Given a manifold $\mathbb{M} \subset \mathbb{R}^m$ of dimension n. A *gnomonic atlas* $\{\boldsymbol{\psi}_0, \Sigma\}$ for a manifold $\mathbb{M}$ is composed of the inverse chart $\boldsymbol{\psi}_0$ and a set of symmetries Σ of $\mathbb{R}^m$ such that

$$\mathbb{M} = \bigcup_{\sigma_i \in \Sigma} \sigma_i \circ \boldsymbol{\psi}_0([-1,1]^n) \tag{15}$$

[2] **M. Godard**, L. Jaulin, D. Massé, "Inner and outer approximation of the image of a set by a nonlinear function", *International Journal of Approximate Reasoning*, 2025.

Image of the boundary of a set: $\mathbf{g}_i$ functions



Construction of the $\mathbf{g}_i$ functions

If $\{\boldsymbol{\psi}_0, \Sigma\}$ is a gnomonic atlas of $\partial\mathbb{X}$, we denote

$$\mathbf{g}_i = \mathbf{f} \circ \sigma_i \circ \boldsymbol{\psi}_0 \tag{16}$$

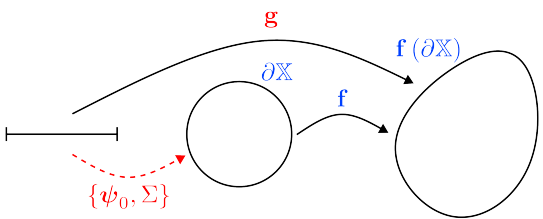
We then have

$$\mathbf{f}\left(\partial\mathbb{X}\right) = \bigcup_i \mathbf{g}_i\left([-1,1]^{m-1}\right) \tag{17}$$

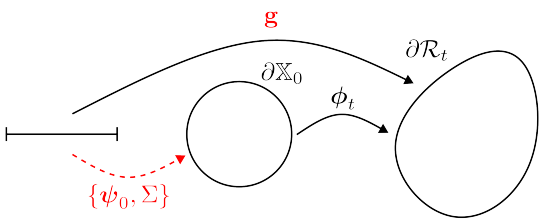
Each $\mathbf{g}_i$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, $n \leq m$. We have

$$\mathbf{f}\left(\partial\mathbb{X}\right) \subset \bigcup_i \langle \mathbf{g}_i \rangle([-1,1]^{m-1}) \tag{18}$$

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Consider the IVP

$$\begin{cases} \dot{\mathbf{x}} = \gamma(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{19}$$

The solution of this ODE is the flow function $\boldsymbol{\phi}_t$. If we denote $\mathbf{A}(\mathbf{x}_0, t) = \frac{\partial \boldsymbol{\Phi}_t}{\partial \mathbf{x}_0}(\mathbf{x}_0)$, it satisfies the *variational equation*

$$\dot{\mathbf{A}} = \frac{d\gamma}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{A} \tag{20}$$

with $\mathbf{A}(0) = I$.

Solving the IVP

$$\begin{cases} \dot{\mathbf{x}} = \gamma(\mathbf{x}) \\ \dot{\mathbf{A}} = \frac{d\gamma}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{A} \\ (\mathbf{x}(0), \mathbf{A}(0)) = (\mathbf{x}_0, I) \end{cases} \tag{21}$$

For a given $\mathbf{x}_0 \in \mathbb{R}^n$ will give both $\boldsymbol{\phi}_t(\mathbf{x}_0)$ and its Jacobian. We will use the **CAPD::DynSys** library.

Consider the Brusselator system following the ODE

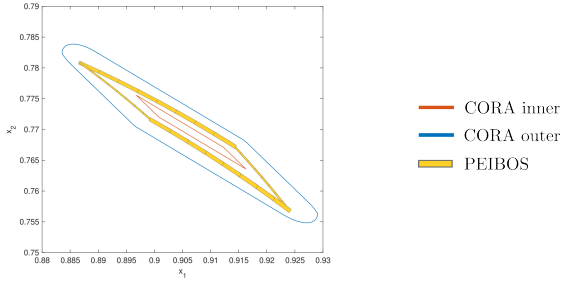
$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 1 - 2x_1 + \frac{3}{2}x_1^2 x_2 \\ x_1 - \frac{3}{2}x_1^2 x_2 \end{pmatrix} \tag{22}$$

We integrate the box $[0.9, 1] \times [0, 0.1]$ over $3s$[3].

[3] CORA example "nonlinear_reachInner_01_brusselator"

Comparison with the CORA library

	Computation time (s)	Langage	Numerical guarantee
CORA	20	Matlab	×
PEIBOS (CODAC & CAPD)	0.18	C++	✓



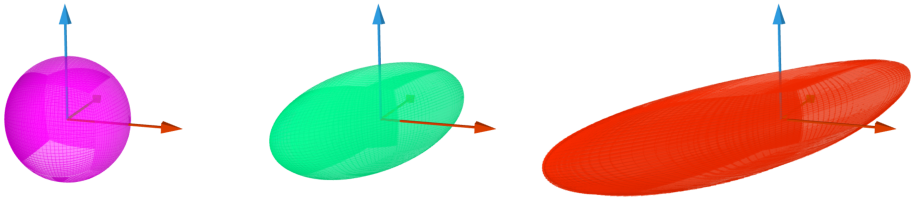
Inner and outer approximations of the brusselator

3D example: Lorenz system

The Lorenz system is defined by :

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} \sigma(x_2 - x_1) \\ \rho x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - \beta x_3 \end{pmatrix}, \sigma = 10, \rho = 28, \beta = \frac{8}{3}$$

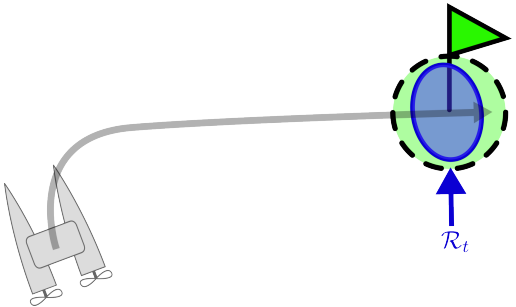
We integrate the unit sphere over $0.1s$



$t = 0s$ $t = 0.05s$ $t = 0.1s$

Integration of the Lorenz system

How to save computation time?



Condition to verify

Plan

Introduction

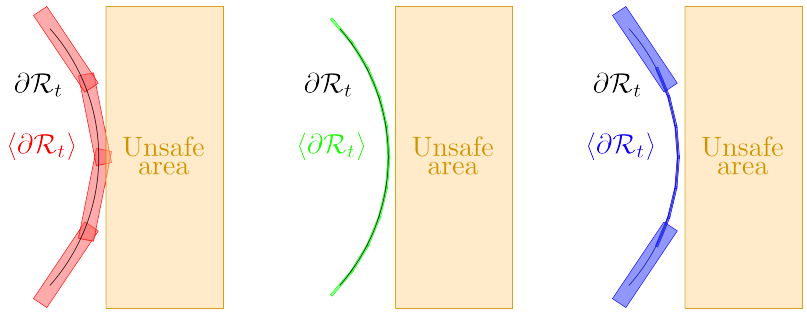
Inner and outer approximation of the image of a set by a nonlinear function

Adaptive parallelepiped enclosure

Inner and outer approximations of the reachable set under disturbances

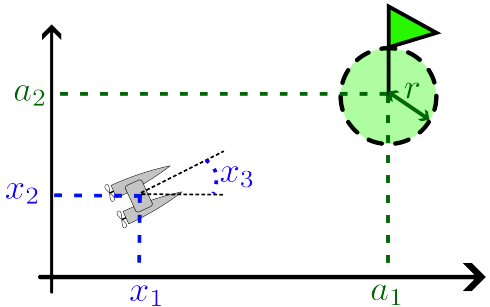
Fake boundary removal

Conclusion



Bad, good and adaptative approximations

Example: area reaching

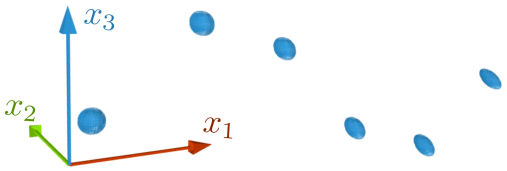


Area reaching of the car

Consider the Dubins car following the ODE

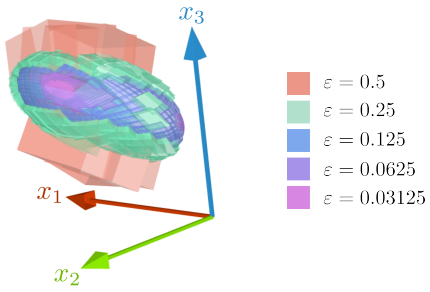
$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{t} \end{pmatrix} = \begin{pmatrix} \cos(x_3) \\ \sin(x_3) \\ \cos(2\pi t) \\ 1 \end{pmatrix} \tag{23}$$

The robot initial set is a ball of radius 0.03.



Reference trajectory with PEIBOS

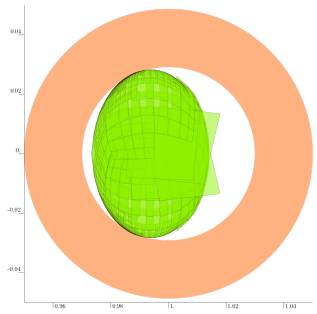
Last time step[4]

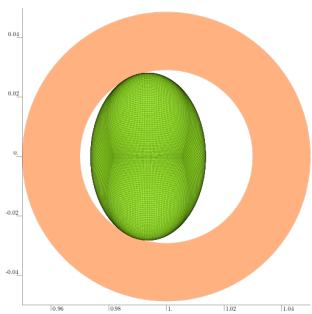


Adaptive enclosure of $\mathcal{R}_1(\partial\mathbb{X}_0)$

[4] **M. Godard**, L. Jaulin, D. Massé, "Constrained adaptive parallelepipedic approximation of the image of a set by a nonlinear function", *International Journal of Approximate Reasoning*, 2026.

Comparison[5]





$\Pi_2\left(\mathcal{R}_1\left(\partial\mathbb{X}_0\right)\right)$

Unsafe area

Adaptive

Classical

Constraint validated in 7 times less time

[5] **M. Godard**, L. Jaulin, D. Massé, "Constrained adaptive parallelepipedic approximation of the image of a set by a nonlinear function", *International Journal of Approximate Reasoning*, 2026.

Plan

Introduction

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Conclusion

- Inner and outer approximations of the reachable set,
- of a disturbed dynamical system,
- in a guaranteed way.

Let us consider an nonlinear dynamical system. It follows an ODE of the form

$$\dot{\mathbf{x}}(t) = \gamma(\mathbf{x}(t)) + \mathbf{w}(t) \tag{24}$$

Both $\mathbb{X}_0$ and $\mathbb{W}$ are compacts. $\partial\mathbb{X}_0$ is a smooth manifold and $\partial\mathbb{W}$ is an ovaloid.

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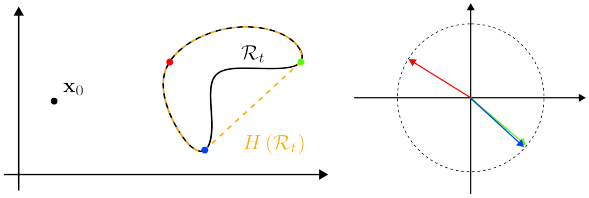
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Origin of the idea[6]

- Starts from an initial state
- Select random initial extremal disturbances
- Make them evolve to reach the extreme states in **a search direction**
- Convexify and inflates the result



Concept of Thomas Lew' reachability analysis

[6] T. Lew, R. Bonalli, M. Pavone, "Exact Characterization of the Convex Hulls of Reachable Sets", *2023 62nd IEEE Conference on Decision and Control (CDC)*.

Modifications[7]

- Start from an initial set,
- Consider all disturbances to avoid the inflation,
- Avoid the convexification.

[7] **M. Godard**, L. Jaulin, D. Massé, "A new interval approach to compute an inner and outer approximation of the reachable set", submitted to *Automatica*.

Modifications[7]

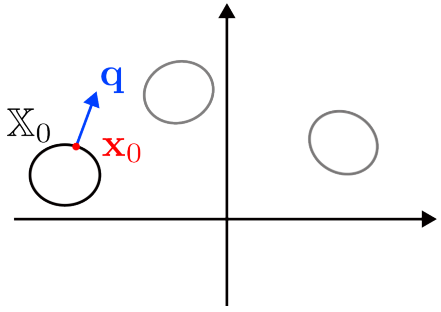
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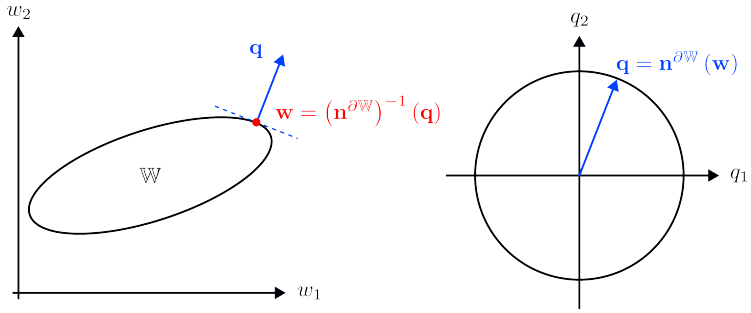
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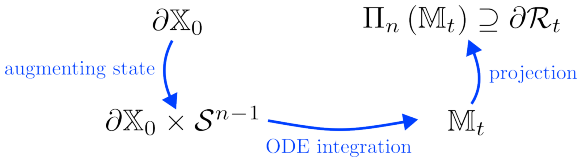


Desired disturbance


w_2
$\mathbf{q}$
$\mathbf{w} = \left(\mathbf{n}^{\partial\mathbb{W}}\right)^{-1}(\mathbf{q})$
$\mathbb{W}$
w_1
q_2
$\mathbf{q} = \mathbf{n}^{\partial\mathbb{W}}(\mathbf{w})$
q_1

Gauss map

$$\begin{cases} \dot{\mathbf{x}} = \boldsymbol{\gamma}(\mathbf{x}) + \left(n^{\partial \mathbb{W}}\right)^{-1}(\mathbf{q}) \\ \dot{\mathbf{q}} = -\mathrm{Proj}_{\mathbf{q}}\left(\frac{d\boldsymbol{\gamma}}{d\mathbf{x}}(\mathbf{x})^{\top}\mathbf{q}\right) \\ \left(\mathbf{x}(0), \left(\mathbf{n}^{\partial \mathbb{W}}\right)^{-1}(\mathbf{q}(0))\right) \in \partial \mathbb{X}_0 \times \partial \mathbb{W} \\ \mathbf{q}(0) = \mathbf{n}^{\partial \mathbb{X}_0}(\mathbf{x}(0)) \end{cases} \tag{25}$$



"Set" point of view

Experimental validation: Video

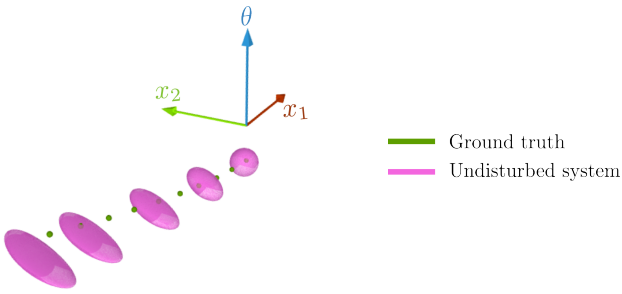


The model of the USV following a heading θ_d is

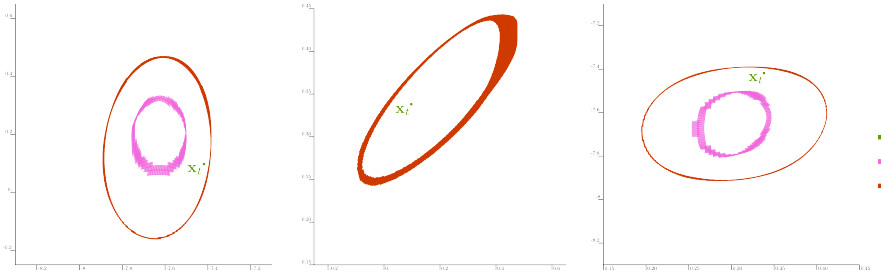
$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -v \cdot \cos(\theta) \\ v \cdot \sin(\theta) \\ K \cdot (\theta_d - \theta) \end{pmatrix}, \; v = 1.48, \; K = 0.2, \; \theta_d = 0.8 \tag{26}$$

$\mathbb{X}_0$ and $\mathbb{W}$ are centered balls of radius 0.15 and 0.06.

Reference trajectory: 3D



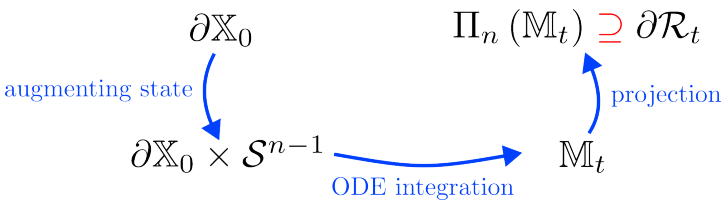
Ground truth (green) and disturbance free model (pink)



(x_1, x_2) cut (x_2, θ) cut (θ, x_1) cut

Cuts and projection of the reachable set at $t = 1.83s$

Non smooth case[8]



"Set" point of view

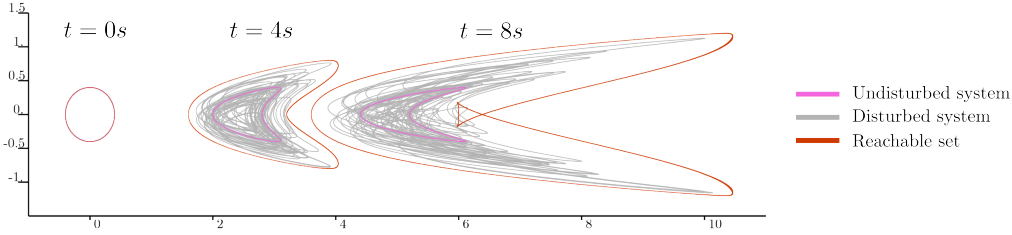
[8] **M. Godard**, L. Jaulin, D. Massé, "A new interval approach to compute an inner and outer approximation of the reachable set", submitted to *Automatica*.

Consider the dynamical system defined by

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_2^2 + 0.6 \\ 0 \end{pmatrix} \tag{27}$$

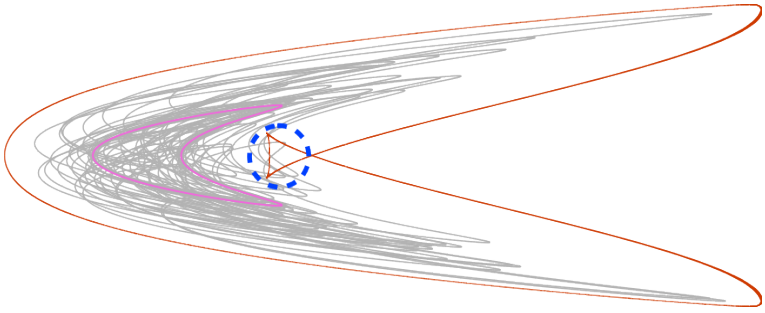
$\mathbb{X}_0$ and $\mathbb{W}$ are centered disk of radius 0.4 and 0.1.

Non smooth example: Result



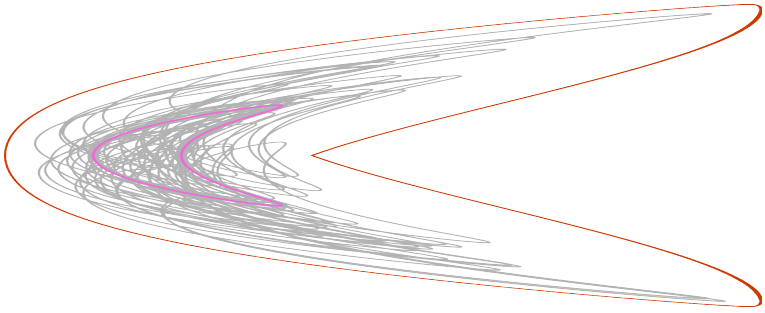
Reference, disturbed and reachable sets through time

Non smooth example: Fake boundary



Reference, disturbed and reachable sets at $8s$

Non smooth example: Expected result



Fake boundary removed

Plan

Introduction

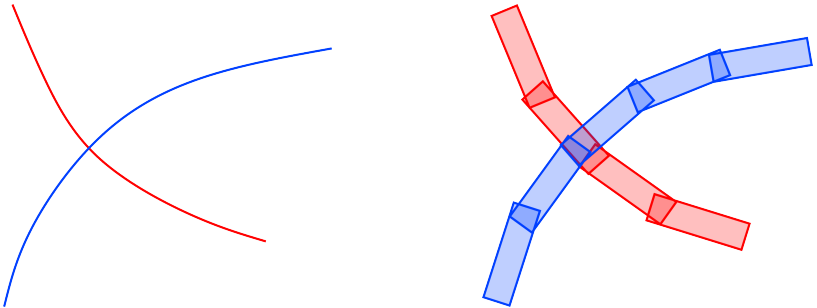
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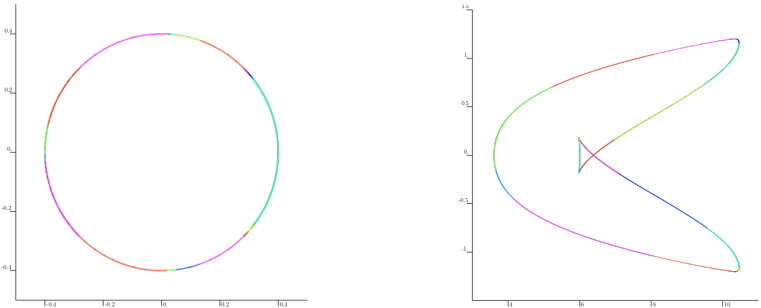
Fake boundary removal

Conclusion



Self intersection in the boundary

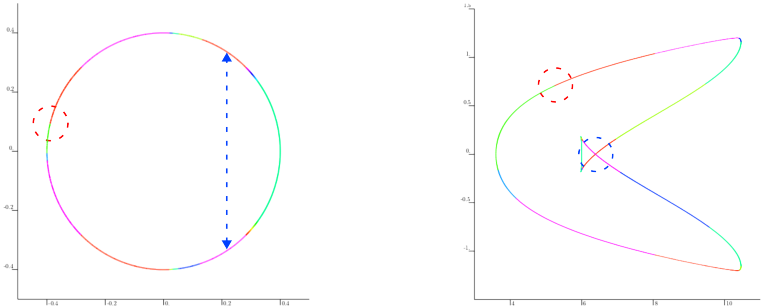
Chain decomposition[9]



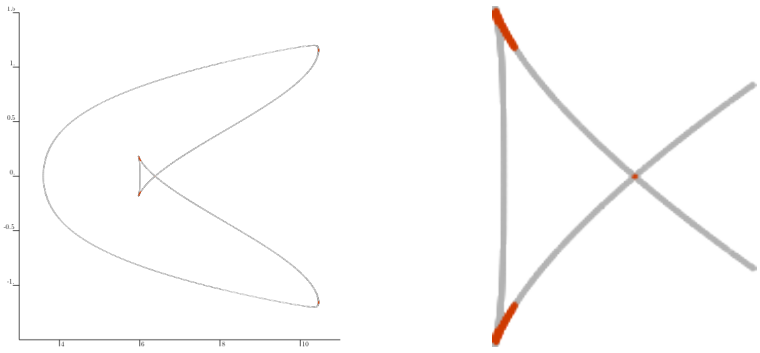
89 chains in the initial (left) and image (right) set

[9] **M. Godard**, L. Jaulin, D. Massé, "Boundary approach to characterize the inner and outer approximation of the image of a disk", *Acta Cybernetica*, 2025.

Chains intersection

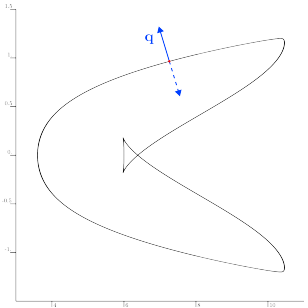


Intersections between chains



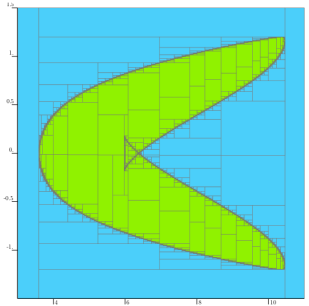
Self intersections

How to classify inner areas?



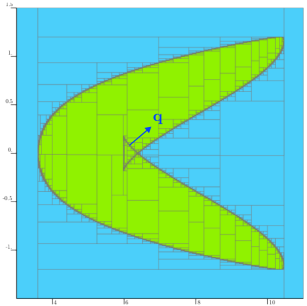
Orthogonal to the surface

Inner and outer approximations



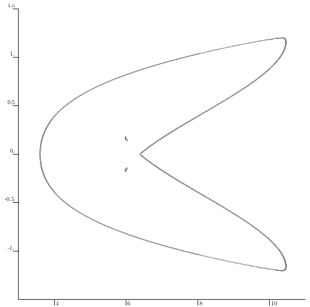
Inner and outer approximations of the reachable set

Fake boundaries detection



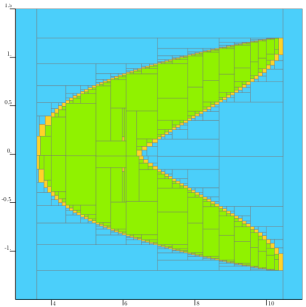
Orthogonal to the surface for fake boundaries

Fake boundaries removed



Part of the fake boundary removed

Improved inner and outer approximations



Inner (green) and outer (green + yellow) approximations of the reachable set

Plan

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- Improve the scalability of the method (different integration scheme...),
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- Diversify the experimental setups (other robots, other mission...),
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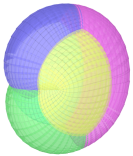
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Contributions: Articles

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- **M. Godard**, L. Jaulin, D. Massé, "Boundary approach to characterize the inner and outer approximation of the image of a disk", *Acta Cybernetica*, 2025.
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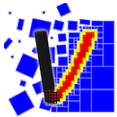
- Created **PEIBOS** (**P**arallelepiped **E**nclosure of the **I**mage of the **BO**undary of a **S**et),
- Multiple developments in **CODAC** including the integration of **PEIBOS**,
- Updates to **VIBes** (viewer),
- All codes are available on a public repository.



PEIBOS (creator)



CODAC (developer)



VIBes (developer)

Contributions to libraries

Conferences and seminars

- Submeeting 2024 and 2026
- JJCR 2024 and 2025
- SWIM 2024 and 2025
- SCAN 2025
- International Online Seminar on Interval Methods in Control Engineering

Thank you for your attention

Libraries and codes

- PEIBOS: https://godardma.github.io/subpages/libs/parallelepiped.html
- CODAC: https://codac.io/
- VIBes: https://github.com/ENSTABretagneRobotics/VIBES
- Codes: https://github.com/godardma/thesis_example

Neighborhood relation

Definition

Let $[\mathbf{x}_i] \in \mathbb{IR}^n$ and $[\mathbf{x}_j] \in \mathbb{IR}^n$ be two boxes. We define the *neighborhood relation* noted R_n between $[\mathbf{x}_i]$ and$[\mathbf{x}_j]$ as :

$$[\mathbf{x}_i] \, R_n \, [\mathbf{x}_j] \Leftrightarrow [\mathbf{x}_i] \cap [\mathbf{x}_j] \neq \emptyset \tag{28}$$

If two boxes are in neighborhood relation, they necessarily intersect.

Chain relation

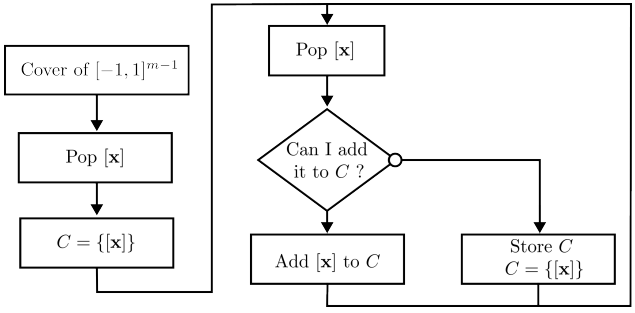
Definition

Let $[\mathbf{x}_i] \in \mathbb{IR}^n$ and $[\mathbf{x}_k] \in \mathbb{IR}^n$ be two boxes and $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$ a C^1 function. $[\mathbf{x}_i]$ and $[\mathbf{x}_k]$ are in *Chain relation for* $\mathbf{g}$, noted $R_{BC,\mathbf{g}}$ is equivalent to :

$$\exists m \in \mathbb{N}, [\mathbf{x}_{j_1}], \ldots, [\mathbf{x}_{j_m}] \in \mathbb{IR}^n, \begin{cases} [\mathbf{x}_i] \, R_n \, [\mathbf{x}_{j_1}] \wedge \cdots \wedge [\mathbf{x}_{j_m}] \, R_n \, [\mathbf{x}_k] \\ \mathbf{g} \text{ is injective on } [\mathbf{x}_i] \cup [\mathbf{x}_{j_1}] \cup \ldots \cup [\mathbf{x}_{j_m}] \cup [\mathbf{x}_k] \end{cases} \tag{29}$$

In our case, the different $\mathbf{g}_i = \boldsymbol{\phi}_t \circ \sigma_i \circ \boldsymbol{\psi}_0$ functions will be studied.

Chain decomposition



Chain decomposition

Thomas Lew article

Exact Characterization of the Convex Hulls of Reachable Sets

Thomas Lew[1], Riccardo Bonalli[2], Marco Pavone[1]

***Abstract*— We study the convex hulls of reachable sets of nonlinear systems with bounded disturbances. Reachable sets play a critical role in control, but remain notoriously challenging to compute, and existing over-approximation tools tend to be conservative or computationally expensive. In this work, we exactly characterize the convex hulls of reachable sets as the convex hulls of solutions of an ordinary differential equation from all possible initial values of the disturbances. This finite-dimensional characterization unlocks a fast sampling-based method to accurately over-approximate reachable sets. We give applications to neural feedback loop analysis and robust model predictive control.**

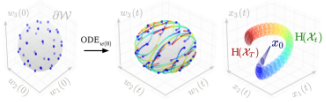


Fig. 1: Theorem 1 states that the convex hulls $\text{H}(\mathcal{X}_t)$ of the reachable sets $\mathcal{X}_t$ can be computed by (a) integrating an augmented ODE (see $\textbf{ODE}_{w(0)}$) for different initial values of the disturbances $w(0) \in \partial\mathcal{W}$, and (b) taking the convex hulls of the resulting extremal state trajectories $x_{w(0)}$.

Reachability method hypothesis

- $\mathbb{X}_0$ is compact and its boundary $\partial\mathbb{X}_0$ is a smooth manifold
- $\mathbb{W}$ is compact and its boundary $\partial\mathbb{W}$ is an ovaloid
- $\mathbf{f}$ is at least C^1, globally Lipschitz and its Jacobian $\frac{d\mathbf{f}}{d\mathbf{x}}$ is Lipschitz
- The disturbances $\mathbf{w} \in L^\infty\left([0, T], \mathbb{W}\right)$ are integrable

Robust under approximate

Robust under-approximations and application to reachability of non-linear control systems with disturbances

Eric Goubault[1] and Sylvie Putot[1]

***Abstract*—We describe a set-based approach, relying on mean-value extensions, for computing guaranteed under-approximations of ranges (or images) of continuously differentiable functions f from $\mathbb{R}^m$ to $\mathbb{R}^n$, including what we call robust ranges, i.e. ranges of functions under adversarial uncertainties. Our method is capable of computing efficiently, at a low computational cost, full n-dimensional subsets of the image of f. As an application, we show how to compute under-approximations of robust reachable sets of non-linear controlled dynamical systems under time-varying uncertainties, which is central to many verification problems in control theory.**

***Index Terms*—Uncertain systems, Computer-aided control design**

approximation problem for robust ranges of functions relies mostly on mean-value theorems, revisiting some work of A. Goldsztejn on modal intervals [2], [3]. With respect to our previous work on the under-approximation of reachable sets for control systems [4], [5], we develop a systematic way to under-approximate the image of vector-valued functions f and no longer just projections of f, without resorting to computer intensive set inversion methods [6]. This is necessary to be able to verify more general properties of control systems. In Section [11] these results are used, combined with a Taylor-based over-approximating reachability analysis, for computing under-approximations of reachable sets of non-linear contin-

$$\mathcal{R}^f_{AE}(t; Z_0, \mathcal{U}) = \{z \in \mathbb{R}^n \mid \forall u_A \in \mathcal{U}_A,\ \exists u_E \in \mathcal{U}_E,\ \exists z_0 \in Z_0,\ z = \varphi_f(t; z_0, u)\} \quad (30)$$

- u_A is “an uncontrollable disturbance”
- u_E is “a controllable input signal”

Hamilton-Jacobi

A Time-Dependent Hamilton–Jacobi Formulation of Reachable Sets for Continuous Dynamic Games

Ian M. Mitchell, *Member, IEEE*, Alexandre M. Bayen, *Member, IEEE*, and Claire J. Tomlin, *Member, IEEE*

***Abstract*—We describe and implement an algorithm for computing the set of reachable states of a continuous dynamic game. The algorithm is based on a proof that the reachable set is the zero sublevel set of the viscosity solution of a particular time-dependent Hamilton–Jacobi–Isaacs partial differential equation. While alternative techniques for computing the reachable set have been proposed, the differential game formulation allows treatment of nonlinear systems with inputs and uncertain parameters. Because the time-dependent equation's solution is continuous and defined throughout the state space, methods from the level set literature can be used to generate more accurate approximations than are possible for formulations with potentially discontinuous solutions. A numerical implementation of our formulation is described and has been released on the web. Its correctness is verified through a two vehicle, three dimensional collision avoidance example for which an analytic solution is available.**

***Index Terms*—Differential games, Hamilton–Jacobi equations, reachability, verification.**

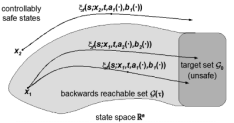


Fig. 1. Target set and backward reachable set. Several trajectories are shown starting at the same time t but from different states x and subject to different input signals $a(\cdot)$ and $b(\cdot)$. Input signal $a(\cdot)$ is chosen to drive the trajectory away from the target set, while input signal $b(\cdot)$ is chosen to drive the trajectory toward the target. The labeling of safe and unsafe states shown is the most common (but not the only possible) interpretation of these sets.

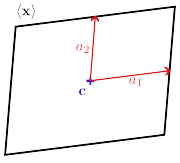
Ian M. Mitchell article

Parallelepiped

Definition

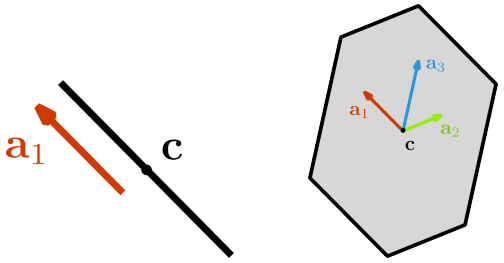
A *parallelepiped* $\langle \mathbf{x} \rangle \subset \mathbb{R}^n$ can be defined by it center $\mathbf{c} \in \mathbb{R}^n$ and its shape matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ as

$$\langle \mathbf{x} \rangle = \mathbf{c} + \mathbf{A} \cdot [-1, 1]^n \tag{31}$$



Parallelepiped in 2D

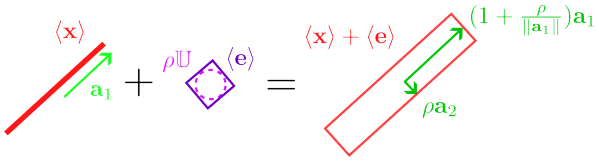
Non square cases



Flat parallelepiped and zonotope

Parallelepiped inflation: Flat case

If the parallelepiped is flat, we first complete its shape matrix with additional generators. They can be chosen arbitrarily, or by using the null space of the flat shape matrix.



Parallelepiped inflation : Flat case

Parallelepiped inflation: Algorithm

Algorithm 5 Parallelepiped inflation

Input: $\langle \mathbf{y} \rangle, \rho$ such that $\langle \mathbf{y} \rangle = \mathbf{A} \cdot \begin{pmatrix} [-e_1, e_1] \\ \vdots \\ [-e_n, e_n] \end{pmatrix}$

Output: $\langle \mathbf{z} \rangle$

function Inflate($\langle \mathbf{y} \rangle, \rho$)
    $\mathbf{K} \leftarrow \text{Kernel}(\mathbf{A})$
    $\tilde{\mathbf{A}} \leftarrow [\mathbf{A} \mid \mathbf{K}]$
    $\mathbf{Q} \leftarrow (\tilde{\mathbf{A}}^T \tilde{\mathbf{A}})^{-1}$
    $\mathbf{M} \leftarrow 0^{m \times m}$
    for $i = 1$ to m **do**
        $\mathbf{M}(i,i) \leftarrow \rho \sqrt{\mathbf{Q}(i,i)}$
    end for
    for $i = 1$ to n **do**
        $\mathbf{M}(i,i) \leftarrow \mathbf{M}(i,i) + e_i$
    end for
    $\langle \mathbf{z} \rangle \leftarrow \tilde{\mathbf{A}} \cdot \mathbf{M} \cdot [-1, 1]^m$
end function

Inflation algorithm

Convergence order: Definition

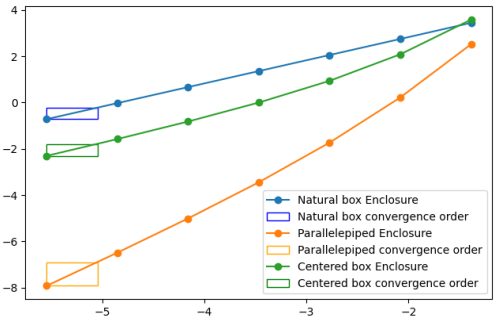
Definition

Given $\mathbf{g} : \mathbb{R}^n \mapsto \mathbb{R}^m$ a C^1 function with $n < m$. The inclusion function $\langle \mathbf{g} \rangle$ converges with an order k if, for any tiny box $[\mathbf{x}]$, we have

$$\text{vol}(\langle \mathbf{g} \rangle([\mathbf{x}])) = \varepsilon^m \varepsilon^{k(m-n)} \tag{32}$$

where $\varepsilon = O(w([\mathbf{x}]))$. The *order of convergence* of $\langle \mathbf{g} \rangle$ is the largest k such that $\langle \mathbf{g} \rangle$ converges with an order k.

Convergence order: Example



Convergence order

Boundary approach

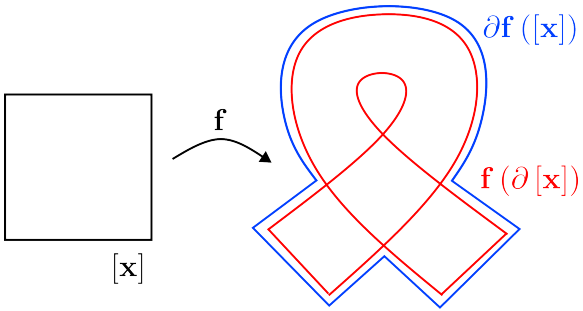
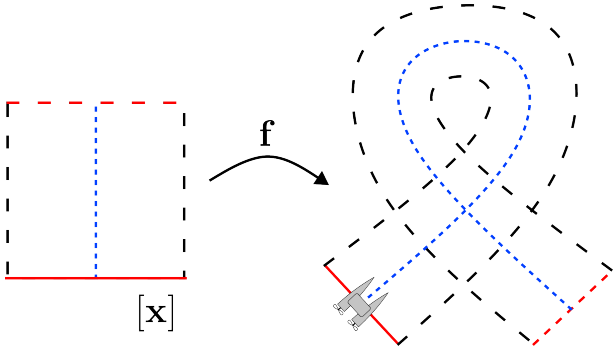


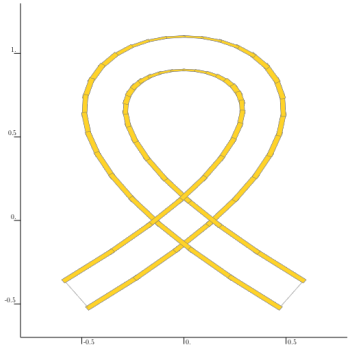
Image of the boundary

Example : Explored area



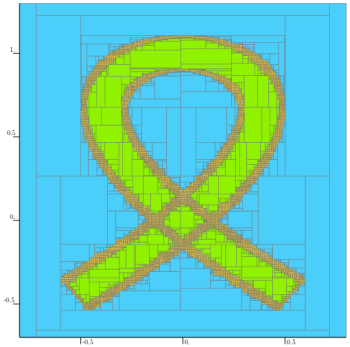
Explored area as a direct image computation

Example : Parallelepiped enclosure



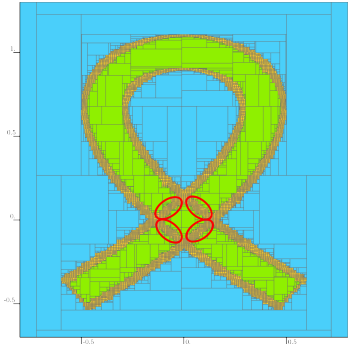
Output of the PEIBOS algorithm

Example : Inner and outer approximations



Inner and outer approximations of the explored area ($\varepsilon = 0.0625$)

Example : fake boundaries



Fake boundaries

Constraint and contractor

Definition

A *constraint* $\mathcal{L}$ is a boolean function defined by :

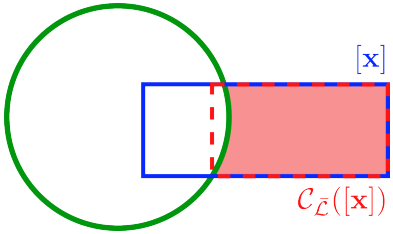
$$\mathcal{L}(\mathbf{x}) = \begin{cases} 1 & \text{if the constraint is satified} \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

Definition

A *contractor* $\mathcal{C}_{\mathcal{L}}$ associated with the constraint $\mathcal{L}$ is a function which satisfies the following properties

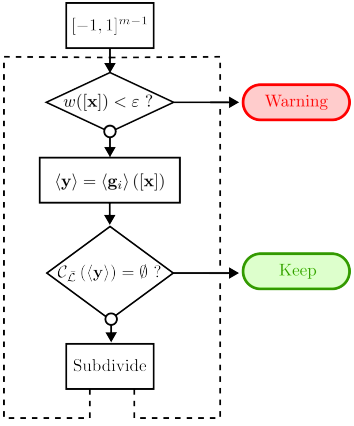
- $\forall [\mathbf{x}] \in \mathbb{IR}^n$, $\mathcal{C}_{\mathcal{L}}([\mathbf{x}]) \subseteq [\mathbf{x}]$ (Contractance)
- $\forall \mathbf{x} \in [\mathbf{x}]$, $\mathcal{L}(\mathbf{x}) = 1 \implies \mathbf{x} \in \mathcal{C}_{\mathcal{L}}([\mathbf{x}])$ (Consistency)

Contractor on the complementary

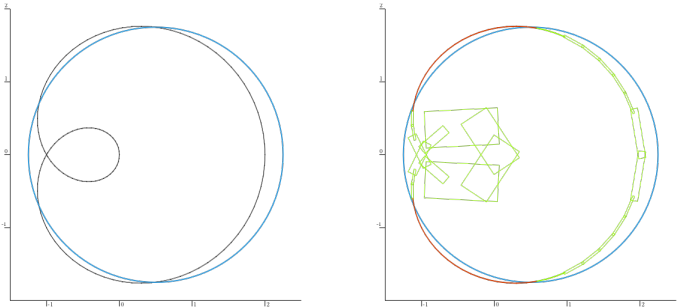


Contractor on the complementary

Adaptive parallepiped enclosure

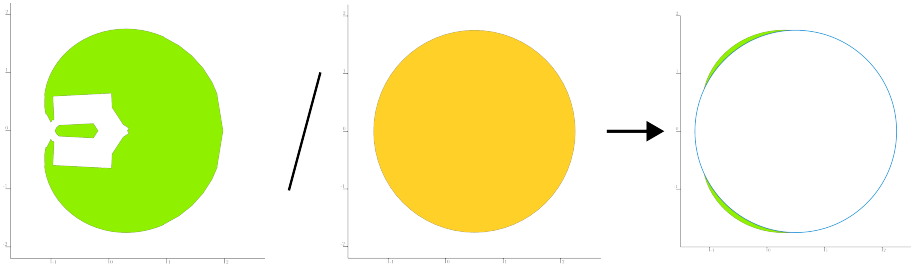


Adaptive example



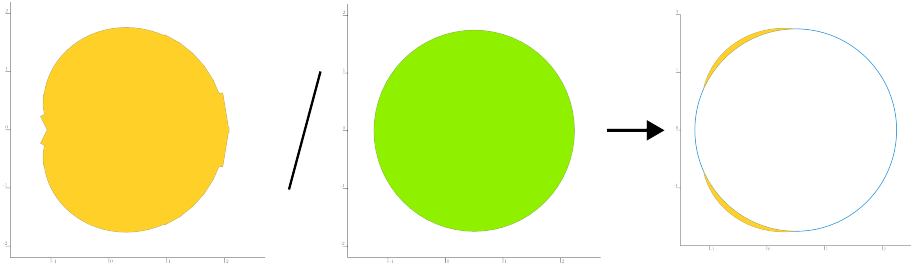
Reference and adaptive

Adaptive: Inner



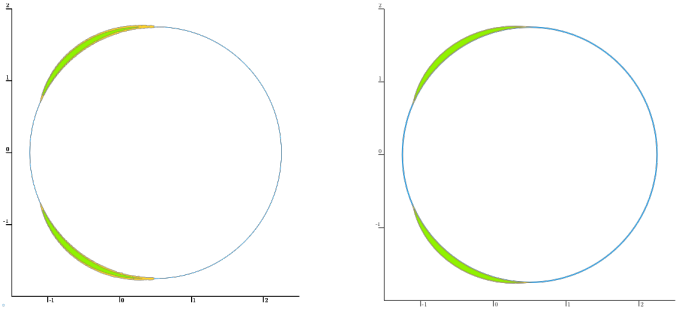
Inner approximation

Adaptive: outer



Outer approximation

Comparison with inversion



Inversion and adaptive parallelepiped approaches

ARCH-COMP

CORA. (Matthias Althoff, Mark Wetzlinger) The tool *COntinuous Reachability Analyzer* (CORA) [6, 7] realizes techniques for reachability analysis with a special focus on developing scalable solutions for verifying hybrid systems with nonlinear continuous dynamics and/or nonlinear differential-algebraic equations. A further focus is on considering uncertain parameters and system inputs. Due to the modular design of CORA, much functionality can be used for other purposes that require resource-efficient representations of multi-dimensional sets and operations on them. CORA is implemented as an object-oriented MATLAB code. The modular design of CORA makes it possible to use the capabilities of the various set representations for other purposes besides reachability analysis. While CORA uses verified algorithms, it does not consider rounding errors since the main focus of the toolbox is the fast prototyping of new reachability algorithms and concepts, and for this purpose the effect of rounding errors is usually negligible. CORA is available at cora.in.tum.de.

"it [CORA] does not consider rounding errors"

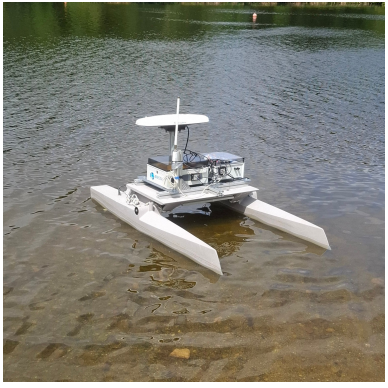
CORA interval implementation[10]

used algorithm is described in detail. Since CORA does not consider rounding errors of floating point numbers, the first version of the interval arithmetic implemented in CORA also neglects this effect. Rounding effects are difficult to consider for transcendental functions due to the *table maker's dilemma* [8, Sec. 2.1.4], [20]. Computation of rounding errors for transcendental functions has been taken special care of in INTLAB, but details of those algorithms, which are specific to each function evaluation, are not published to the best knowledge of the authors. In the first version, we also restrict ourselves to intervals of real numbers and exclude intervals of complex numbers.

"CORA does not consider rounding errors"

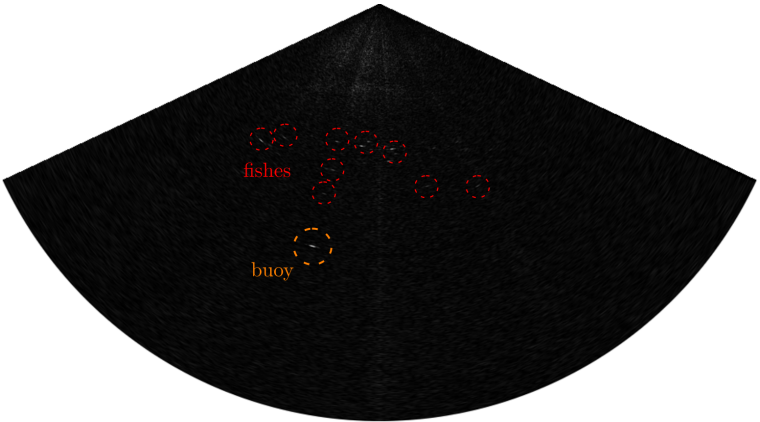
[10] M. Althoff and D. Grebenyuk, "Implementation of interval arithmetic in CORA 2016," in Proc. of the 3rd International Workshop on Applied Verification for Continuous and Hybrid Systems, 2016, p. 91–105

Hélios



Hélios USV

Sonar detection of buoy



Sonar detections

Serment [En]

In the presence of my peers,

With the completion of my doctorate in robotics, in my quest for knowledge, i have carried out demanding research, demonstrated intellectual rigour, ethical reflection, and respect for the principles of research integrity. As i pursue my professional carreer, whatever my chosen field, i pledge, to the greatest of my ability, to continue to maintain integrity in my relationship to knowledge, to my methods and to my results .

Serment [Fr]

En présence de mes pairs,

Parvenu(e) à l'issue de mon doctorat en robotique, et ayant ainsi pratiqué, dans ma quête du savoir, l'exercice d'une recherche scientifique exigeante, en cultivant la rigueur intellectuelle, la réflexivité éthique et dans le respect des principes de l'intégrité scientifique, je m'engage, pour ce qui dépendra de moi, dans la suite de ma carrière professionnelle quel qu'en soit le secteur ou le domaine d'activité, à maintenir une conduite intègre dans mon rapport au savoir, mes méthodes et mes résultats.